A super-Brownian motion with a locally infinite catalytic mass

Klaus Fleischmann[1] and Carl Mueller[2,*]

[1]Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D–10117 Berlin, Germany (e-mail: fleischmann@wias-berlin.de)

[2]Department of Mathematics, University of Rochester, Rochester, NY 14620, USA (e-mail: cmlr@troi.rochester.edu)

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Abstract

A super-Brownian motion X in $\mathbb{R}$ with "hyperbolic" branching rate $\varrho_2(b) = 1/b^2$, $b \in \mathbb{R}$, is constructed, which symbolically could be described by the formal stochastic equation

$$\mathrm{d}X_t = \tfrac{1}{2}\,\Delta X_t\,\mathrm{d}t + \sqrt{2\varrho_2 X_t}\,\mathrm{d}W_t\,, \qquad t > 0, \tag{1}$$

(with a space-time white noise $\mathrm{d}W$).

Starting at $X_0 = \delta_a$, $a \neq 0$, this superprocess X will never hit the catalytic center: There is an increasing sequence of Brownian stopping times τ_n strictly smaller than the hitting time of 0 such that with probability one Dynkin's stopped measures X_{τ_n} vanish except for finitely many n.

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1 Introduction and results

1.1 Motivation and purpose

A continuous *super-Brownian motion (SBM)* $X = \{X_t;\ t \geq 0\}$ in $\mathbb{R}$ with *branching rate* $\varrho(b) \geq 0,\ b \in \mathbb{R}$, can *heuristically* be thought of as follows:

Many particles with small mass move independently on the line $\mathbb{R}$ according to standard Brownian motions. Additionally each particle at position b may die with a large rate proportional to $\varrho(b)$, or it may split with the same rate into two particles situated again at b which continue to evolve independently and according to the same rules. If we now denote by $X_t(B)$ the mass at time t in the Borel set B, then the measure X_t describes the cloud of mass at time t. Although X_t is not integer-valued (since the mass of particles is asymptotically small), it is useful to interpret $X_t(\mathrm{d}b)$ as the mass of all *particles* situated in b at time t. (For background, we refer to Dawson [Daw93].)

In the simplest case, the branching rate ϱ is a constant. But it may also vary in space and even in time (*varying medium*). For instance, consider the case $\varrho(b) = (2\varepsilon)^{-1}\mathbf{1}\{|b-c| \leq \varepsilon\},\ \ b \in \mathbb{R}$, which means that branching is allowed only if particles are in a small neighborhood of a fixed point $c \in \mathbb{R}$, and then the rate is huge. Even the limiting model as $\varepsilon \to 0$ makes sense non-trivially (in this one-dimensional situation). Then formally one can write $\varrho = \delta_c$ (Dirac δ-function at c), and speak of a single *point catalyst* situated at c; see [DF94, DFLM95, FL95, Dyn95] or the surveys [Fle94, DFL95]. In Dynkin's [Dyn91a] terminology, in this case the branching phenomenon of the approaching particles is governed by the Brownian local time at c.

More generally, ϱ may be a fairly general non-negative Schwartz distribution, that is, the generalized derivative of a measure, which we denote by the same symbol ϱ; see [DF91, DFR91, DF95, DLM95]. (Or ϱ could additionally be time-dependent, for instance a continuous super-Brownian motion, in which case the catalytic masses themselves suffer a branching mechanism; see [DF96].) But so far as we know, a common assumption is that the generalized function ϱ is *locally integrable,* as in the δ-function case $\varrho = \delta_c$; that is, ϱ corresponds to a locally finite measure.

Our *first purpose* in this paper is to demonstrate that a super-Brownian motion X with a *locally infinite* branching rate measure ϱ may make sense (Theorem 3). Then of course the question arises whether such a branching measure-valued process has qualitatively new properties. Intuitively one can expect that X has significantly more extinction features in the area where the branching rate measure is locally unbounded.

Indeed, our *second aim* is to exhibit the following new effect. We consider a particular branching rate ϱ (as ϱ_2 in (1)) which has a sufficiently infinite (accumulated) catalytic mass around a center c. Then, starting X at $X_0 = \delta_a$, $a \neq c$, the branching population will never hit c. Actually, the infinite catalytic mass around c will *kill* all the hidden particles *before* they reach c (Theorem 4).

The *figure* shows a *simulation* of equation (1) with initial condition $X_0(b) \equiv 1$, but with the singular branching rate ϱ_2 replaced by the truncated rate $\varrho_2 \wedge K$ with $K = 10^4$. Large fluctuations around the catalytic center $c = 0$ are clearly exhibited, whereas extinction at $c = 0$ is not apparent, because of the truncation.

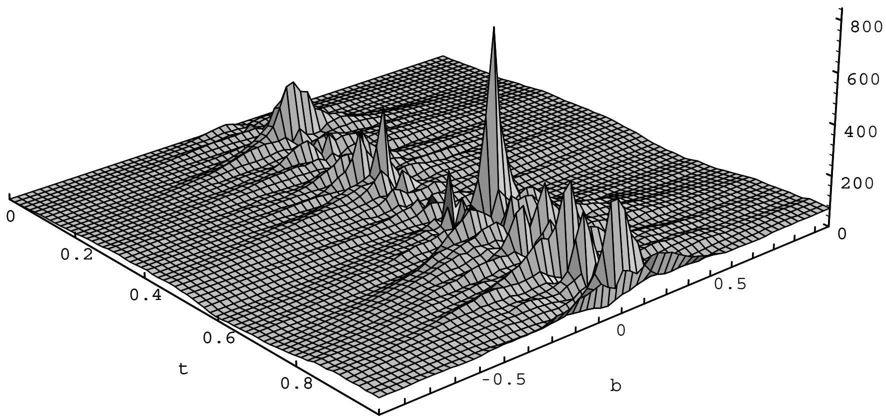

Figure: Simulation of a solution to equation (1)

1.2 Existence in the case of a locally infinite catalytic mass

We completely concentrate on the *one-dimensional* case, where, by the way, all the super-Brownian motions known so far have absolutely continuous states ([DFR91]). To warm up we will start with the *single point catalytic model* $\varrho = \theta\delta_c$, where $\theta \geq 0$ is an additional weight of the point catalyst, which we let tend to infinity. But it turns out that in this case the *limiting* model is degenerate: The limiting infinite point catalyst will only instantaneously kill the mass, that is, no mass is born. This results in the heat flow with absorption, i.e. the randomness of the model disappears (Proposition 7 at p.11).

Going away from this degenerate situation, our *main model* is based on the following *branching rate*

$$\varrho_\sigma(b) := \frac{\theta}{|b-c|^\sigma}, \qquad b \in \mathbb{R}, \tag{2}$$

for $c \in \mathbb{R}$, $\theta > 0$ and $\sigma \geq 0$ fixed. If $0 \leq \sigma < 1$, we get a special case of a model constructed in [DF91, DFR91], since here in particular the catalytic measure $\varrho_\sigma(b)\,\mathrm{d}b$ is locally finite. On the other hand, by a limitation of our methods, as a rule we exclude large σ. For convenience, we introduce the following notation.

Definition 1 (hyperbolic branching rate) Under $1 \leq \sigma \leq 2$, the branching rate ϱ_σ of (2) is called *hyperbolic.* Moreover, we distinguish between a *moderate* hyperbolic branching rate if $1 \le \sigma < 2$, and a *critical* one if $\sigma = 2$. Analogously, the related super-Brownian motions X (to be constructed in § 4.2) are also called *hyperbolic, moderate*, etc., in the respective cases. $\diamond$

Remark 2 The name *critical* hyperbolic branching rate is motivated by the fact that under $\sigma = 2$ (and $c = 0$) the related log-Laplace equation (33) admits *self-similar* solutions; see Remark 15 at p.19. $\diamond$

As opposed to the previously mentioned model of a point catalyst with a limiting infinite weight, in the present case outside of the catalytic center c now we have a non-degenerate critical branching mechanism allowing a proper stochastic process.

On the other hand, *intuitively speaking*, under $1 \le \sigma \le 2$, the infinite catalytic mass around the hyperbolic pole will again kill the Brownian particles eventually arriving at c. Thus, the underlying motion law should "effectively" be the Brownian motion W^c *killed at* c (non-conservative Markov process), and we will indeed finally arrive at W^c as "underlying" motion process. Note also that this heuristic picture of Brownian particles killed at c says that at c no birth of mass will occur. In particular, the usual criticality of the branching mechanism will "effectively" be violated at c. This also makes transparent that the total mass process $t \mapsto X_t(\mathsf{I\!R})$ should *not longer* be a martingale, as opposed to the usual critical super-Brownian motions with a locally finite branching rate measure.

Here is our first theorem (a more precise description will be given with Theorem 19 at p.23):

Theorem 3 (hyperbolic SBM X) *Assume* $1 \le \sigma \le 2$.

(a) (existence) *There exists a non-degenerate (finite measure-valued) superprocess* $X = \{X_t\,;\ t \ge 0\}$ *in* $\mathsf{I\!R}$ *with hyperbolic branching rate* ϱ_σ *and Brownian motion killed at c as motion law.*

(b) (total mass process) *The total mass process* $X(\mathsf{I\!R}) = \{X_t(\mathsf{I\!R});\ t \ge 0\}$ *is a supermartingale but no longer a martingale. Its variance is finite if and only if* $\sigma < 2$.

(c) (convergence) *In the sense of convergence of all finite-dimensional distributions,* X *is the limit in law as* $K \to \infty$ *of the super-Brownian motion* X^K *in* $\mathsf{I\!R}$ *with truncated branching rate* $\varrho_\sigma \wedge K$.

For the critical exponent $\sigma = 2$, the limit process X is a superprocess which, as far as we know, does *not* yet appear in the literature, in spite of all the serious efforts to construct the most general superprocess; see Dynkin [Dyn94], Dynkin and Kuznetsov [DKS94], and Leduc [Led95], to mention only a few recent sources. In fact, although the branching mechanism is "binary critical" everywhere in the new phase space $\mathsf{I\!R}\backslash\{c\}$, the limit process X does not have a finite variance (under $\sigma = 2$).

In the terminology of Dynkin [Dyn94, § 1.3.1], X is a *sub*critical superprocess with motion law given by the Brownian motion W^c killed at c. But the branching rate ϱ_σ is *un*bounded, and the additive functional $K(\mathrm{d}r) = \varrho_\sigma(W_r^c)\,\mathrm{d}r$ of W^c has *infinite* characteristic under $\sigma = 2$. That is, the expectation of $\int_0^t K(\mathrm{d}r)$ is infinite, [Dyn94, (3.2.2)]. (Indeed, the characteristic of K is finite if and only if $\sigma < 2$; to see this, use Lemma 5 at p.8.) Therefore, under $\sigma = 2$, our process X does not fit into the framework of [Dyn94, DKS94, Led95]. On the other hand, for $1 \leq \sigma < 2$, our process X might be considered as a special case of known processes, but our construction for the critical exponent $\sigma = 2$ covers $1 \leq \sigma < 2$, so we do not need such reduction.

1.3 Strong killing in the critical case $\sigma = 2$

It can be expected that the hyperbolic SBM X dies in finite time. Here is a more important question: Is it possible that under $\sigma = 2$ *all* the hidden Brownian particles *die before they reach* c? The *main result* of the paper will be a *positive* answer to this question. To formulate it, we make use of the "stopped measures" X_τ in the sense of Dynkin [Dyn91a, Dyn91b]: Intuitively, if τ is a (finite) stopping time of Brownian motion, then X_τ describes the cloud of all the branching Brownian particles in their moments τ.

Theorem 4 (strong killing in the case of a critical ϱ_2) *Assume that X is a super-Brownian motion in $\mathbb{R}$ with a critical hyperbolic branching rate ϱ_2 and with starting measure $X_0 = \delta_a$, $a \neq c$. Then there exists an increasing sequence of Brownian stopping times τ_n which are strictly smaller than the Brownian (first) hitting time τ^c of the catalytic center c, such that with probability one the stopped measures X_{τ_n} vanish except for finitely many n.*

Consequently, here all population mass dies before it reaches the catalytic center c, that is, the superprocess X *does not hit* c. Of course, at this stage the formulation of this theorem is a bit vague. Anyway, a precise description will be given with Theorem 23 at p.28.

1.4 Tools and outline

An essential tool for our approach is the *historical* superprocess $\widetilde{X}$ related to X, we now roughly want to describe (for background in the locally finite branching measure case, see [DP91] or [Dyn91b]). To this aim, the measures $X_t(\mathrm{d}a)$ on $\mathbb{R}$ are thought as projections of measures $\widetilde{X}_t(\mathrm{d}w)$, where w are continuous functions on $\mathbb{R}_+$ stopped at time t. Heuristically, each particle hidden in the cloud of mass X_t, with position a at time t, is now additionally equipped with the path $w : [0,t] \to \mathbb{R}$ with $w_t = a$. This path gives the past history of the particle.

As a further refinement we switch to *"stopped" historical superprocesses:* Hidden particles are stopped at any stopping time $\tau < \tau^c$ of Brownian motion, instead of t, resulting in a random measure $\widetilde{X}_\tau(\mathrm{d}w)$ defined on paths w stopped at τ. Consequently, $\widetilde{X}_\tau$ describes the "mass distribution of the cloud traced back" from the point of view of the Brownian random moment τ.

There is a simple *heuristic argument* for our main result. Assume $c = 0$. Consider the total local time L^a at $a > 0$ of a Brownian path killed at 0. For small a, the expected value of L^a is of order a. Hence, on average the "amount of branching" should be given by the expected value of $\int_0^\infty \mathrm{d}a\, L^a \varrho_\sigma(a)$, which is bounded below by *const* $\int_0^1 \mathrm{d}a\, a^{1-\sigma}$. Thus an infinite (averaged) amount of branching occurs exactly in the critical case $\sigma = 2$. But an infinite amount of branching means that the process dies before it hits 0.

We feel that the probabilistic method used in this paper is of some independent interest, and furthermore we will use it in a future paper [DFM96] to study the die-out of super-Brownian motion with various kind of random catalysts. However, the referee has suggested an alternative, *analytic approach* to our main result (Theorem 4). This approach uses the fact that in the case $c = 0$, for $0 < p < 1$ fixed, the function $a \mapsto u_p(a) := \theta^{-1} p^{p+1}(1+a)a^{-p}$, $a > 0$, solves the elliptic differential inequality $\frac{1}{2}\,\Delta u \leq \varrho_2 u^2$. In other words, u_p is a *supersolution* to $\frac{1}{2}\,\Delta u = \varrho_2 u^2$ that blows up at 0 and ∞. Then, adapting Iscoe's [Isc88] method to the present case, with $\mathrm{e}^{-u_p(a)}$ we would get a lower bound on the probability that the *range* of X (starting with $X_0 = \delta_a$) does not hit the catalytic center. As $p \downarrow 0$, this bound tends to 1.

The *outline of the paper* is as follows. In the next section the case of a single infinite point catalyst is investigated. In Section 3, we deal with the log-Laplace equation which we treat in a Feynman-Kac approach. The construction of the hyperbolic SBM is provided in Section 4, and indeed in a setting of historical superprocesses. The proof of the strong killing in the case of a critical hyperbolic branching rate then follows in the final section.

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2 Single point-catalytic model: degeneration

As announced, in this section we discuss the degenerate case of a single, infinite point catalyst.

2.1 Preliminaries: Some notation

We adopt the following conventions. If E is a topological space then subsets of E will be equipped with the induced topology. Products of topological spaces will be endowed with the product topology. Measures on a topological space E will be defined on the Borel σ-algebra (generated by the open subsets of E). A measure m on E with $m(E\backslash E')=0$ for some measurable E', that is, m is concentrated on E', will also be regarded as a measure on E' (and conversely).

If E_1 and E_2 are topological spaces, let $\mathcal{B}[E_1, E_2]$ denote the space of all *measurable* maps $f: E_1 \mapsto E_2$. Write $b\mathcal{B}[E_1, E_2]$ for the subset of all *bounded* functions. As a rule, $b\mathcal{B}[E_1, E_2]$ is equipped with the topology of *bounded pointwise convergence*. Sometimes we also work with the supremum norm $\|f\|_\infty := \sup\{\|f(e_1)\|;\ e_1 \in E_1\}$ of *uniform convergence*, where $\|\cdot\|$ is the norm in E_2. By $\mathcal{C}[E_1, E_2]$ and $b\mathcal{C}[E_1, E_2]$ we denote the spaces of all *continuous* f in $\mathcal{B}[E_1, E_2]$ or $b\mathcal{B}[E_1, E_2]$, respectively.

$\mathcal{M}[E_1]$ refers to the set of all finite (non-negative) measures on a Polish space E_1, endowed with the topology of *weak* convergence. The pairing $\langle \mu, \varphi\rangle$ denotes the integral $\int\mu(\mathrm{d}e_1)\,\varphi(e_1)$, $\mu \in \mathcal{M}[E_1]$, $\varphi \in \mathcal{B}[E_1, \mathbb{R}]$ (if it exists).

Write simply $\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}$ etc., if the respective spaces E_i coincide with the real line $\mathbb{R}$. The lower index $+$ on the symbol of a set will always refer to the subset of all of its non-negative members.

2.2 Brownian motion killed at c

Let $W = \left[W, \Pi_a, a \in \mathbb{R}\right]$ denote the *standard Brownian motion* in $\mathbb{R}$ starting at time 0. We use the symbol Π_a also to describe the expectation with respect to the law Π_a for the process starting at a, and proceed similarly in related situations. For instance, for $\mu \in \mathcal{M}$, define $\Pi_\mu f(W) := \int\mu(\mathrm{d}a)\int\Pi_a(\mathrm{d}w)\, f(w)$, for reasonable functionals f.

Denote by $S = \{S_t;\, t \geq 0\}$ the Brownian semigroup acting on $\mathcal{B}_+$, and by p the related (continuous) Brownian transition density function,

$$\mathrm{p}_t(a,b) = \mathrm{p}_t(b-a) := \tfrac{1}{\sqrt{2\pi\, t}}\,\exp\!\left[-\,\tfrac{(b-a)^2}{2t}\right], \qquad t>0, \quad a,b\in\mathbb{R},$$

(fundamental solutions of the heat equation).

Recall that τ^c refers to the *hitting time of* c of the Brownian motion W. Set

$$S_t^c\varphi(a) := \Pi_a \mathbf{1}\{\tau^c > t\}\varphi(W_t), \qquad t>0, \quad a\in\mathbb{R}, \quad \varphi\in\mathcal{B}_+, \tag{3}$$

for a fixed $c \in \mathbb{R}$, and

$$\langle S_t^c\mu, \varphi\rangle := \langle \mu, S_t^c\varphi\rangle, \qquad t>0, \quad \mu\in\mathcal{M}, \quad \varphi\in\mathcal{B}_+ .$$

We call $\{S_t^c;\ t>0\}$ the *semigroup of Brownian motion killed at* c, and the "dual" $\{S_t^c\mu;\ t>0\}$ the *heat flow with absorption at* c, starting with μ. This is justified by the fact that restricting to non-negative measurable functions φ defined only on $\{a \neq c\}$ and adding the identity operator, we actually get the semigroup of a non-conservative Markov process on $\{a \neq c\}$, the Brownian motion killed at c. Also, by the reflection principle of Brownian motion,

$$\mathrm{p}_t^c(t,a,b) := \mathbf{1}\Big\{(a-c)(b-c)>0\Big\}\Big[\mathrm{p}_t(a-b)-\mathrm{p}_t(a+b-2c)\Big], \qquad (4)$$

$t>0,\ a,b\in\mathbb{R}$, yields the transition kernels related to S^c. Note that $\mathrm{p}_t^c(a,b)$ is (strictly) positive if and only if $(a-c)(b-c)>0$.

For convenience, we now explain the following simple but useful transition density estimate we need in later sections.

Lemma 5 (bounds for p^c) *Let $c=0$. Then there are constants $0<c_1<c_2$ such that*

$$c_1\exp\big[-\tfrac{2a^2}{t}\big] \leq \tfrac{t^{3/2}}{ab}\,\mathrm{p}_t^c(a,b) \leq c_2\exp\big[-\tfrac{a^2}{8t}\big], \qquad t>0, \quad 0<b<\tfrac{a}{2}.$$

Proof To get the upper bound, use the mean value theorem and $\frac{a}{2}\leq a+\vartheta b\leq \frac{3}{2}a$ for $|\vartheta|\leq 1$. On the other hand, to obtain the lower bound, apply the elementary inequality $\mathrm{e}^{-x}-\mathrm{e}^{-y}\geq (y-x)\mathrm{e}^{-y},\ y\geq x\geq 0$, and $a+b\leq 2a$. ■

As a consequence of the upper bound in Lemma 5 we get the following first moment bound involving the hyperbolic branching rate ϱ_σ of (2):

Lemma 6 (uniform finiteness) *Under* $1\leq\sigma\leq 2$,

$$\alpha_T := \sup\nolimits_{a\in\mathsf{R}}\,\Pi_a\,\mathbf{1}\{\tau^c>T\}\int_0^T \mathrm{d}r\ \varrho_\sigma(W_r)<+\infty, \qquad T>0, \qquad (5)$$

and $\alpha_T = T^{1-\sigma/2}\alpha_1$.

(Note that this expectation differs from the "characteristic" mentioned at p.5, since, reading (5) in terms of Brownian motion W^c killed at c, survival is required even at the *terminal* time T.)

Proof The claimed T–dependence of α_T immediately follows from Brownian scaling. Hence for the uniform finiteness proof, which will be provided in several steps, we may set $T=1$.

Step 1° Without loss of generality, we put $c=0$ and restrict our attention to $a>0$. Changing the order of expectation and integration, and using the Markov property at time r, we look at

$$\sup_{a>0}\int_{(0,1)}\mathrm{d}r\int_{(0,\infty)}\mathrm{d}b\ \mathrm{p}_r^c(a,b)\,b^{-\sigma}\ \Pi_b\big\{\tau^c>1-r\big\} \qquad (6)$$

instead of the expression in (5). We may additionally restrict the internal integral to $0 < b < 1$ (finiteness in the opposite case is trivial). For the internal probability expression we have

$$\Pi_b\{\tau^c > 1-r\} = \int_{1-r}^{\infty} \mathrm{d}s \ \tfrac{b}{\sqrt{2\pi\, s^3}} \exp\bigl[-\tfrac{b^2}{2s}\bigr]. \tag{7}$$

By substitution, this can be continued with

$$= \text{const} \int_{2(1-r)/b^2}^{\infty} \mathrm{d}s \ s^{-3/2} e^{-1/s} \ \le \ \text{const} \ \min\Bigl(1,\, b/\sqrt{1-r}\Bigr)$$

(where *const* always denotes a constant, which may change from one to another expression). Inserting into (6), we are left with showing the finiteness of

$$\sup_{a>0} \int_0^1 \mathrm{d}r \int_0^1 \mathrm{d}b \ \mathrm{p}_r^c(a,b)\, b^{-\sigma} \ \min\Bigl(1,\, b/\sqrt{1-r}\Bigr). \tag{8}$$

Step 2° Next we additionally assume that $r \le \frac{1}{2}$ (the opposite case will be dealt with in Step 3° below). Then we may replace the minimum expression in (8) by b, to arrive at

$$\sup_{a>0} \int_0^{1/2} \mathrm{d}r \int_0^1 \mathrm{d}b \ \mathrm{p}_r^c(a,b)\, b^{1-\sigma}\,. \tag{9}$$

Now we distinguish between three cases concerning the variable b in the internal integral:

(i) $b \le \frac{a}{2}$ Then by Lemma 5 we get the bound

$$\sup_{a>0} \int_0^{1/2} \mathrm{d}r \int_0^1 \mathrm{d}b \ b^{2-\sigma}\, a\, r^{-3/2}\, \mathrm{e}^{-a^2/8r}$$

which (up to a multiplicative constant) equals

$$\sup_{a>0} \int_0^{1/2} \mathrm{d}r \ a\, r^{-3/2}\, \mathrm{e}^{-a^2/8r} \ \le \ \int_0^{\infty} \mathrm{d}r \ r^{-3/2}\, \mathrm{e}^{-1/r} \ < +\infty. \tag{10}$$

(ii) $a < \frac{b}{2}$ Again we apply Lemma 5 (using the symmetry $\mathrm{p}_r^c(a,b) = \mathrm{p}_r^c(b,a)$) to find the bound

$$\text{const} \sup_{0<a\le 1/2} \int_0^{1/2} \mathrm{d}r \int_{2a}^1 \mathrm{d}b \ b^{1-\sigma}\, a\, b\, r^{-3/2}\, \mathrm{e}^{-b^2/8r}.$$

Interchange the order of integration, use the uniform finiteness (10) (with b instead of a), and note that

$$\sup_{0<a\leq 1/2} a \int_{2a}^{1} \mathrm{d}b \; b^{1-\sigma} < +\infty$$

(distinguishing between $\sigma = 2$ which leads to $\sup_{0<a\leq 1/2} a|\log a|$, and $\sigma < 2$ which is still easier).

(iii) $\frac{a}{2} \leq b \leq 2a$ Here simply use $\mathrm{p}_r^c(a,b) \leq \mathrm{const}\, r^{-1/2}$ which results in a finite $\mathrm{d}r$–integral, and $b^{1-\sigma} \leq \mathrm{const}\, a^{1-\sigma}$, to arrive at the bound

$$\mathrm{const} \sup_{0<a\leq 2} a^{1-\sigma} \int_{a/2}^{1\wedge 2a} \mathrm{d}b \leq \mathrm{const} \sup_{0<a\leq 2} a^{2-\sigma} < +\infty.$$

Altogether, (9) is finite.

Step 3° Finally, assume in (8) additionally that $r > \frac{1}{2}$. Moreover, if in addition $b \geq \sqrt{1-r}$, then we simply drop the minimum expression in (8), use that $\mathrm{p}_r^c(a,b) \leq \mathrm{const}$, and that

$$\int_{\sqrt{1-r}}^{1} \mathrm{d}b \; b^{-\sigma} = \mathrm{const}\, (1-r)^{\frac{1}{2}-\frac{\sigma}{2}}$$

is $\mathrm{d}r$–integrable (for $r \leq 1$). Therefore we are left with the case $b < \sqrt{1-r}$, and have to demonstrate that

$$\sup_{a>0} \int_{1/2}^{1} \mathrm{d}r \;\; (1-r)^{-1/2} \int_{0}^{\sqrt{1-r}} \mathrm{d}b \; \mathrm{p}_r^c(a,b)\, b^{1-\sigma} < +\infty. \tag{11}$$

For this we again distinguish between three cases.

(j) $b \leq \frac{a}{2}$ Then by Lemma 5 we get the finite bound

$$\int_{1/2}^{1} \mathrm{d}r \;\; (1-r)^{-1/2} \int_{0}^{1} \mathrm{d}b \; b^{2-\sigma} \sup_{a>0} a\, \mathrm{e}^{-a^2/8}$$

(up to a constant).

(jj) $a < \frac{b}{2}$ Again by Lemma 5 we find the bound

$$\mathrm{const} \sup_{0<a\leq 1/2} \int_{1/2}^{1} \mathrm{d}r \;\; (1-r)^{-1/2} \int_{2a}^{1} \mathrm{d}b \; b^{2-\sigma} a < +\infty.$$

(jjj) $\frac{a}{2} \leq b \leq 2a$ Here simply replace the density $\mathrm{p}_r^c(a,b)$ by a constant, use once more the inequality $b^{1-\sigma} \leq \mathrm{const}\, a^{1-\sigma}$, to arrive at the bound

$$\mathrm{const} \sup_{0<a\leq 2} \int_{1/2}^{1} \mathrm{d}r \;\; (1-r)^{-1/2}\, a^{1-\sigma} \int_{a/2}^{2a} \mathrm{d}b \leq \mathrm{const} \sup_{0<a\leq 2} a^{2-\sigma} < +\infty.$$

Hence, (11) is correct, finishing the proof altogether. ∎

2.3 Single point-catalytic super-Brownian motion

Consider the *continuous single point catalytic super-Brownian motion* $X^\theta = \{X_t^\theta;\, t > 0\}$ with branching rate $\theta\delta_c$, where $c \in \mathbb{R}$ and, for the moment, $\theta \geq 0$ is fixed. That is, X^θ is the continuous superprocess related to the formal log-Laplace equation

$$\tfrac{\partial}{\partial t}\, v_\theta \;=\; \tfrac{1}{2}\,\Delta v_\theta - \theta\,\delta_c\, v_\theta^2 . \tag{12}$$

Consequently, the critical branching phenomenon is restricted to the location c of the point catalyst whereas outside c only the heat flow acts. As usual for superprocesses, the connection to (12) is given by a Laplace transition functional:

$$E_\mu \exp\left\langle X_t^\theta, -\varphi\right\rangle = \exp\left\langle \mu,\, -v_\theta(t,\cdot)\right\rangle, \qquad \mu \in \mathcal{M}, \quad t > 0, \quad \varphi \in b\mathcal{B}_+ ,$$

where v_θ solves (12) in a mild sense with initial condition $v_\theta(0+,\cdot) = \varphi$. For a detailed exposition of this point-catalytic SBM X^θ we refer to [DF94] or [FL95]. Note that μ serves as the initial measure X_0^θ of X^θ although we formally excluded X_0^θ from the notation X^θ for the sake of a simpler formulation of the following proposition.

Proposition 7 (degeneration) *As $\theta \to \infty$ and in the sense of weak convergence of all finite-dimensional distributions* (fdd)*, the single point-catalytic super-Brownian motion X^θ degenerates to the heat flow with absorption at c :*

$$X^\theta \overset{\text{fdd}}{\Longrightarrow} X^\infty := \left\{S_t^c X_0\,;\, t > 0\right\} \quad \text{as} \quad \theta \to \infty.$$

Roughly speaking, if the catalytic mass θ of the point catalyst will be infinite, then all population mass which arrives at the catalyst will immediately be killed, and no branching occurs anymore in the model. In particular, $X_t^\infty \equiv 0$ for $t > 0$, provided that $X_0\big(\mathbb{R}\backslash\{c\}\big) = 0$, that is if the initial measure X_0 is concentrated at c.

This proposition will be proved in the next subsection.

Remark 8 (Γ–stable catalysts) Let Γ denote the *stable* random measure on $\mathbb{R}$ with index $0 < \gamma < 1$ determined by its Laplace functional

$$E \exp\langle \Gamma, -\varphi\rangle \;=\; \exp\Big[-\int \mathrm{d}b\;\varphi^\gamma(b)\Big], \qquad \varphi \in b\mathcal{B}_+ .$$

For the moment, consider the super-Brownian motion X^θ with branching rate $\theta\Gamma$, $\theta \geq 0$. That is, X^θ is the superprocess related to the formal equation

$$\tfrac{\partial}{\partial t}\, v_\theta \;=\; \tfrac{1}{2}\,\Delta v_\theta - \theta\,\Gamma\, v_\theta^2, \qquad v_\theta(0+) = \varphi \geq 0 \tag{13}$$

(see [DF91]). Then Proposition 7 suggests that X_t^θ weakly converges (as $\theta \to \infty$) to the *heat flow with absorption at* Γ, which should degenerate to $X_t^\infty \equiv 0$ for $t > 0$, since the atoms of Γ (point catalysts) are *dense* in $\mathbb{R}$. In terms of the related equation (13) this should mean that $v_\theta(t,a) \underset{\theta\to\infty}{\longrightarrow} 0,\;\; t > 0,\; a \in \mathbb{R}$. $\diamond$

2.4 Proof of the degeneration proposition

We start the *Proof of Proposition* 7 by recalling first the following approach [FL95] to the continuous single point catalytic super-Brownian motion X^θ with a finite initial state X_0^θ which w.l.o.g. can be assumed to be a deterministic measure $\mu \in \mathcal{M}$. Start by introducing the transition densities q of a standard *stable subordinator* with index $\frac{1}{2}$ on $\mathbb{R}_+$:

$$\mathrm{q}_s(a,b) = \mathrm{q}_s(b-a) := \frac{s}{\sqrt{2\pi\,(b-a)^3}}\,\exp\bigl[-\tfrac{s^2}{2(b-a)}\bigr], \tag{14}$$

$s > 0,\ \ 0 \le a < b$. Let U^θ denote the related *super-stable subordinator* in $\mathbb{R}_+$ with index $\frac{1}{2}$ and constant branching rate θ. Assume that the initial state $U_0 := U_0^\theta$ of U^θ is given by

$$U_0(\mathrm{d}r) := \mathrm{d}r \int \mu(\mathrm{d}b)\ \mathrm{q}_{|c-b|}(r), \qquad r \ge 0, \tag{15}$$

with q defined in (14) and using the convention $\mathrm{d}r\,\mathrm{q}_0(r) = \delta_0(\mathrm{d}r)$. That is, U_0 is the "law" of the hitting time τ^c of c of a Brownian motion starting at time 0 "distributed" according to the initial measure μ of X^θ (the latter has to be constructed). In particular, $U_0 = \delta_0$ if $\mu = \delta_c$.

Now let $V_\infty^\theta := \int_0^\infty \mathrm{d}s\, U_s^\theta$ denote the *total occupation time* (measure) related to the measure-valued process U^θ. Then the single point catalytic super-Brownian motion X^θ can be defined by

$$X_t^\theta(\mathrm{d}b) \ :=\ S_t^c\mu(\mathrm{d}b) + \left(\int_{[0,t)} V_\infty^\theta(\mathrm{d}s)\ \mathrm{q}_{|b-c|}(t-s)\right)\mathrm{d}b, \qquad t>0; \tag{16}$$

see formula (16) in [FL95].

To understand this statement, recall that $\mathrm{q}_{|b|}(s)$ is also Itô's Brownian excursion from 0 density at time s at $|b|$. Hence, X^θ results from two parts. Namely first from the initial mass described by μ which propagates according to the heat flow $S^c\mu$ with absorption at c. The second contribution comes from some randomly created mass which starts at time s from the catalyst with the amount $V_\infty^\theta(\mathrm{d}s)$ and spreads deterministically away according to the mentioned Itô's excursion density. In particular, V_∞^θ yields the occupation density measure of X^θ at c (super-local time measure at c).

Based on this representation formula (16) and by the Markov property of X^θ, for the proof of Proposition 7 it suffices to show that the total mass $V_\infty^\theta(\mathbb{R})$ of the total occupation measure V_∞^θ converges to 0 in distribution as $\theta \to \infty$. But from the definition of the super-stable subordinator U^θ follows that

$$t \mapsto \int_0^t \mathrm{d}s\ U_s^\theta(\mathbb{R}) =: V_t^\theta(\mathbb{R}), \qquad t \ge 0,$$

is the occupation time process related to *Feller's critical branching diffusion* with branching rate θ, starting with $U_0(\mathbb{R}) = \mu(\mathbb{R})$ (recall (15)). That is,

$$E \exp\big[-\lambda V_t^\theta(\mathbb{R})\big] = \exp\big[-\mu(\mathbb{R})\, u_\theta(t)\big], \qquad \lambda \geq 0,$$

where u_θ is the solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\theta(t) = -\theta\, u_\theta^2(t) + \lambda \quad \text{with} \quad u_\theta(0) = 0.$$

But this equation can explicitly be solved, yielding

$$u_\theta(t) = \sqrt{\lambda/\theta}\, \tanh\big[t\sqrt{\lambda\theta}\big] \underset{t\to\infty}{\longrightarrow} \sqrt{\lambda/\theta} \underset{\theta\to\infty}{\longrightarrow} 0.$$

Therefore the $\frac{1}{2}$-stable law of $V_\infty^\theta(\mathbb{R})$ has the scaling parameter $\mu(\mathbb{R})/\sqrt{\theta}$ tending to 0, hence it converges weakly to the Dirac measure δ_0 as $\theta \to \infty$. Hence $V_\infty^\theta(\mathbb{R})$ tends to 0 in distribution as $\theta \to \infty$, finishing the proof of Proposition 7. ■

3 Analytical tool: Feynman-Kac equation

As a preparation for the construction of the hyperbolic SBM X in $\mathbb{R}$ as claimed in the existence Theorem 3, in this section we want to introduce our "main analytical tool" for this: a Feynman-Kac equation. For convenience, here we restrict our attention to a fixed finite time interval $I := [0, T]$, $T \geq 0$. Since we actually need the *historical* superprocess $\widetilde{X}$ related to X, and since this process is a time-*in*homogeneous process, it is convenient to work with a *backward* and *historical* setting from the beginning.

3.1 Preliminaries: Terminology and spaces

We start by introducing some terminology. If A, B are sets and $a \mapsto B^a$ is a map of A into the set of all subsets of B, then we write

$$A \widehat{\times} B^\bullet := \big\{[a, b];\ a \in A,\ b \in B^a\big\} = \textstyle\bigcup_{a\in A} \{a\} \times B^a \tag{17}$$

for the *graph* of this map. Note that $A \widehat{\times} B^\bullet \subseteq A \times B$.

To each path w in the Banach space $\mathbf{C} := \mathcal{C}[I, \mathbb{R}]$, and $t \in I = [0, T]$, we associate the corresponding *stopped path* w^t by setting $w_s^t := w_{t\wedge s}$, $s \in I$. That is, the path is held constant after time t. The set of all *stopped paths* $w^t = \{w_s^t;\ s \in I\}$ is denoted by $\mathbf{C}^t$, getting (for t fixed) a closed subspace of $\mathbf{C}$. Note that $\mathbf{C}^s \subseteq \mathbf{C}^t$ if $s \leq t$, that $\mathbf{C}^T = \mathbf{C}$, and that $\mathbf{C}^0$ can be identified with $\mathbb{R}$, whereas $\mathbf{C}^t$ could also be considered as $\mathcal{C}\big[[0, t], \mathbb{R}\big]$.

To each path w in $\mathbf{C}$, we can also associate the corresponding *stopped path trajectory* $\widetilde{w}$ by setting: $\widetilde{w}_t := w^t$, $t \in I$. Note that $\widetilde{w}$ is a mapping of I into $\mathbf{C}$.

Since, for $0 \le s \le t \le T$,

$$\|\widetilde{w}_t - \widetilde{w}_s\|_\infty = \|w^t - w^s\|_\infty = \sup_{s \le r \le t} \big|w_r - w_s\big| \underset{t-s \to 0}{\longrightarrow} 0,$$

$\widetilde{w}$ actually belongs to the closed subspace

$$\widetilde{\mathbf{C}}(I) := \Big\{\omega \in \mathcal{C}[I, \mathbf{C}];\ \omega_t \in \mathbf{C}^t,\ t \in I\Big\} \tag{18}$$

of the Banach space $\mathcal{C}[I, \mathbf{C}]$. Moreover,

$$\|\widetilde{v} - \widetilde{w}\|_\infty = \sup_{t \in I} \|\widetilde{v}_t - \widetilde{w}_t\|_\infty = \|v^T - w^T\|_\infty = \|v - w\|_\infty, \qquad v, w \in \mathbf{C},$$

hence $w \mapsto \widetilde{w}$ maps $\mathbf{C}$ continuously into the space $\widetilde{\mathbf{C}}(I)$ of stopped path trajectories. Note also that $\widetilde{\mathbf{C}}(I) \subseteq I \widehat{\times} \mathbf{C}^\bullet$ where the latter is a closed subset of $I \times \mathbf{C}$.

3.2 Brownian path processes $\widetilde{W}$ and $\widetilde{W}^c$

Recall that Π_a denotes the distribution of a standard Brownian path W starting at $W_0 = a$. We now regard it as a probability law on $\mathbf{C} = \mathcal{C}[I, \mathbb{R}]$. Then applying the map $w \mapsto \widetilde{w}$ from the previous subsection to W, we get the so-called *Brownian path process* $\widetilde{W} = \big[\widetilde{W}, \widetilde{\Pi}_{s,w}\,,\ s \in I,\ w \in \mathbf{C}^s\big]$ which is a time-inhomogeneous strong Markov process. In other words, at time s we start with a path $w = \widetilde{W}_s$ stopped at time s, and let a path trajectory $\big\{\widetilde{W}_t\,;\ t \in [s, T]\big\}$ evolve with law $\widetilde{\Pi}_{s,w}$ determined by a Brownian path $\{W_t\,;\ s \le t \le T\}$ starting at time s at w_s. We may and will regard $\widetilde{\Pi}_{s,w}$ as a probability law on $\widetilde{\mathbf{C}}\big([s, T]\big)$ (recall (18)).

The *semigroup* of $\widetilde{W}$ will be denoted by $\widetilde{S} = \big\{\widetilde{S}_{s,t}\,;\ 0 \le s \le t \le T\big\}$,

$$\widetilde{S}_{s,t}\,\varphi\,(w) := \widetilde{\Pi}_{s,w}\,\varphi(\widetilde{W}_t), \quad 0 \le s \le t \le T,\ w \in \mathbf{C}^s,\ \varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}], \tag{19}$$

and the related *generator* by $\widetilde{A} = \big\{\widetilde{A}_s\,;\ s \in I\big\}$,

$$\widetilde{A}_s \psi(w) = \lim_{h \downarrow 0} h^{-1}\big[\widetilde{S}_{s-h,s}\,\psi\,(w^{s-h}) - \psi(w)\big], \qquad w \in \mathbf{C}^s,$$

$\psi \in \mathcal{D}(\widetilde{A})$ (that is $\psi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}]$ such that the limit exists).

Analogously, we introduce the *standard Brownian motion killed at* c :

$$W^c = \big[W^c,\, \Pi^c_a\,,\ a \in \mathbb{R}\big],$$

and the related *Brownian path process killed at* c :

$$\widetilde{W}^c = \Big[\widetilde{W}^c,\, \widetilde{\Pi}^c_{s,w}\,,\ s \in I,\ w \in \mathbf{C}^s\Big].$$

Here the Π_a^c are *sub*probability laws satisfying, in particular,

$$\Pi_a^c(W_t^c \in \cdot\,) = \Pi_a\big(\tau^c > t,\ W_t \in \cdot\,\big), \qquad t \in I,\ a \in \mathbb{R}, \tag{20}$$

(where τ^c is the hitting time of the catalytic center c). Here writing the symbol W_t^c tacitly means that W^c is still alive at time t. Analogously,

$$\widetilde{\Pi}_{s,w}^c\big(\widetilde{W}_t^c \in \cdot\,\big) = \widetilde{\Pi}_{s,w}\big(\tau^c > t,\ \widetilde{W}_t \in \cdot\,\big), \quad 0 \le s \le t \le T,\ w \in \mathbf{C}^s. \tag{21}$$

Recall that S^c is the semigroup related to the Brownian motion W^c killed at c, introduced in (3). Denote by $\widetilde{S}^c = \big\{\widetilde{S}_{s,t}^c;\ 0 \le s < t \le T\big\}$ the semigroup of $\widetilde{W}^c$.

As in the case of Brownian motion (introduced in the beginning of § 2.2), we use notations as

$$\widetilde{\Pi}_{s,\mu}\,\varphi\big(\widetilde{W}_t\big) = \int \mu(\mathrm{d}w)\,\widetilde{\Pi}_{s,w}\,\varphi\big(\widetilde{W}_t\big), \tag{22}$$

$0 \le s \le t \le T,\ \mu \in \mathcal{M}(\mathbf{C}^s),\ \varphi \in b\mathcal{B}[\mathbf{C}^t, \mathbb{R}]$.

Of course, from $\widetilde{W}$ and $\widetilde{W}^c$ we can gain back W and W^c by projection. For instance, $W_t := (\widetilde{W}_t)_t$, which will repeatedly be used.

3.3 Truncated equation

Fix a constant $K > 1$ and consider the *truncated rate function* $\varrho_\sigma \wedge K$, where ϱ_σ is the hyperbolic branching rate from (2) (recall that $1 \le \sigma \le 2$).

For $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$ fixed, let $V^K\varphi := v_K$ denote the unique element in $b\mathcal{B}\big[I \widehat{\times} \mathbf{C}^\bullet, \mathbb{R}_+\big]$ (recall (17)), which solves the following non-linear equation *("truncated" log-Laplace equation)*

$$v_K(s, \omega_s) = \widetilde{\Pi}_{s,\omega_s}\Big[\varphi(\widetilde{W}_T) - \int_s^T \mathrm{d}r\ (\varrho_\sigma \wedge K)(W_r)\ v_K^2\big(r, \widetilde{W}_r\big)\Big], \tag{23}$$

$[s, \omega_s] \in I \widehat{\times} \mathbf{C}^\bullet$ (cf. e.g. Dynkin [Dyn91b, Theorem 1.1]). Here $V^K\varphi := v_K$ continuously depends on φ (in the topology of bounded pointwise convergence). Note that by a formal differentiation using the semigroup $\widetilde{S}$ of $\widetilde{W}$ with generator $\widetilde{A}$, from this integral equation we get the partial differential equation

$$\left.\begin{array}{c} -\frac{\partial}{\partial s}\, v_K(s, \omega_s) \ = \ \widetilde{A}_s v_K(s, \omega_s) - (\varrho_\sigma \wedge K)\big((\omega_s)_s\big)\, v_K^2\,(s, \omega_s), \\ \text{with terminal condition } v_K(T, \omega_T) = \varphi(\omega_T). \end{array}\right\} \tag{24}$$

(Here $-\frac{\partial}{\partial s}\, v_K(s, \omega_s) = \lim_{h \downarrow 0}\ h^{-1}\big[v_K\big(s-h, (\omega_s)^{s-h}\big) - v_K(s, \omega_s)\big]$.) Moreover, if φ belongs to the domain $\mathcal{D}(\widetilde{A})$ of $\widetilde{A}$, then v_K actually solves (24). But then it also uniquely solves the following *"truncated Feynman-Kac equation"* (that is, Feynman-Kac version of (23)):

$$v_K(s,\omega_s) \;=\; \widetilde{\Pi}_{s,\omega_s}\,\varphi(\widetilde{W}_T)\exp\Big[-\int_s^T \mathrm{d}r\;(\varrho_\sigma\wedge K)(W_r)\,v_K\big(r,\widetilde{W}_r\big)\Big], \tag{25}$$

$[s,\omega_s]\in I\widehat{\times}\mathbf{C}^\bullet$ (cf. Dynkin [Dyn94, Theorem 4.2.1]). By dominated convergence, also in this equation we can go back to any $\varphi\in b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$.

3.4 Simplified terminal functions

If the terminal function φ (defined on paths $w\in\mathbf{C}^T=\mathbf{C}$) in the truncated equation only depends on $|w_T-c|$ and even in a non-decreasing way, then the solution $v_K=V^K\varphi$ also has a similar property, which we now want to explain in a lemma. Recall that $K>1$ is fixed.

Lemma 9 (simplified terminal condition) *Assume that $\varphi\in b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$ can be represented as*

$$\varphi(w)=f\big(|w_T-c|\big),\qquad w\in\mathbf{C}=\mathbf{C}^T,$$

with $f\in b\mathcal{B}\big[\mathbb{R}_+,\mathbb{R}_+\big]$ being non-decreasing. Then the (unique) solution v_K $=V^K\varphi$ to the truncated equation (23) or (25) has a representation

$$v_K(s,\omega_s) \;=\; g\big(s,\big|(\omega_s)_s-c\big|\big),\qquad [s,\omega_s]\in I\widehat{\times}\mathbf{C}^\bullet, \tag{26}$$

with $g\in b\mathcal{B}\big[I\times\mathbb{R}_+,\mathbb{R}_+\big]$ being non-decreasing in the second coordinate.

Remark 10 By an abuse of notation, in cases such as in the lemma (and in similar situations), we simply write

$$v_K(s,\omega_s)=v_K\big(s,(\omega_s)_s-c\big)=v_K\big(s,\big|(\omega_s)_s-c\big|\big). \qquad\diamond$$

Proof of Lemma 9 Without loss of generality, we may assume that $c=0$.

1° (*Trotter's product formula*) v_K can be thought of as arising in the following way. Fix $n>>1$ and decompose the interval $[0,T]$ into small pieces of length T/n. Now apply alternatively both terms at the r.h.s. of (24), that is consider separately the pure Brownian path process and the pure quadratic absorption. (That this Trotter's product formula-like procedure converges as $n\to\infty$ to v_K also in the present *non*-linear situation can be seen as follows: Via Laplace transition functionals, as in (36) below, one can switch to the corresponding Markov processes, and for their *linear* semigroups one can apply Trotter's product formula, as e.g. in [EK86, Corollary 1.6.7], to get the desired convergence result.)

For a proof by induction, assume that for some k, $0\le k<n$, at time $s_k:=(n-k)T/n$ a representation as in (26) is given (which is certainly true

for $k=0$). Now it suffices to show that such representation is reproduced at time s_{k+1}, if either only the Brownian path process acts, or only the quadratic absorption is effective. In fact, then the claim follows by taking the limit as $n \to \infty$.

2° (*pure Brownian path process semigroup*) By the semigroup property, we have

$$v_K(s_{k+1}, \omega_{s_{k+1}}) = \widetilde{\Pi}_{s_{k+1}, \omega_{s_{k+1}}} v_K\big(s_k, \widetilde{W}_{s_k}\big).$$

By the induction hypothesis and (26), we may continue with

$$= \Pi_{s_{k+1}, (\omega_{s_{k+1}})_{s_{k+1}}} g\big(s_k, |W_{s_k}|\big).$$

But this expression depends only on $a := \big|(\omega_{s_{k+1}})_{s_{k+1}}\big|$, and, moreover, in a non-decreasing way. To see this monotonicity, use a simple coupling argument. In fact, for fixed $0 \le a_1 < a_2$, consider a pair of coupled reflected standard Brownian motions denoted by $[Z^1, Z^2]$, starting at $[a_1, a_2]$, which evolve independently until they hit each other, and are identical afterwards. Then $Z^1 \le Z^2$, hence $g(Z^1) \le g(Z^2)$ from the assumed monotonicity of g, and the claim follows by taking expectations.

3° (*pure quadratic absorption*) We have to solve (in a mild sense) the equation

$$-\tfrac{\partial}{\partial s} v_K(s, \omega_s) = -(\varrho_\sigma \wedge K)\big((\omega_s)_s\big)\, v_K^2(s, \omega_s), \qquad [s, \omega_s] \in I \widehat{\times} \mathbf{C}^\bullet,$$

at time $s = s_{k+1}$. By the semigroup property of solutions, we may fix here our attention to the terminal condition $v_K(s_k, \omega_{s_k}) = g\big(s_k, |(\omega_{s_k})_{s_k}|\big)$, according to the induction hypothesis. As solution we get

$$v_K\big(s_{k+1}, \omega_{s_{k+1}}\big) = g(s_k, a)\Big[1 + g(s_k, a)(s_k - s_{k+1})\,(\varrho_\sigma \wedge K)(a)\Big]^{-1},$$

with $a := \big|(\omega_{s_{k+1}})_{s_{k+1}}\big|$, which is obviously non-decreasing in a. This finishes the proof. ■

3.5 Constant terminal functions

Lemma 9 is applicable if the terminal function φ is a constant. Then we can complement that lemma by the following result. Recall that $K > 1$ is fixed.

Lemma 11 (temporal monotonicity at the catalytic center) *Assume that $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$ equals the constant $m \ge 0$. Then the solution $v_K = V^K m$ to the truncated equation (23)–(25) has the following property. Fix $s \in [0, T)$. Consider only $\omega_t \in \mathbf{C}^t$ with $(\omega_t)_t = c$ for all $t \in [s, T)$. Then $v_K(t, \omega_t)$ is non-decreasing in $t \in [s, T)$.*

Proof Again we may set $c=0$.

$1°$ (*reformulation*) Fix $s<T$, and consider only those $[t,\omega_t] \in [s,T)\widehat{\times}\mathbf{C}^\bullet$ such that $(\omega_t)_t \equiv a$ on $[s,T)$ for some $a \in \mathbb{R}$. By Lemma 9 we may write $v_K(t,\omega_t) = v_K(t,a)$, and from (24) we get

$$-\frac{\partial}{\partial t} v_K(t,a) \;=\; \frac{1}{2}\Delta v_K(t,a) - (\varrho_\sigma \wedge K)(a)\, v_K^2\,(t,a), \qquad s \le t < T,\;\; a \in \mathbb{R},$$

with constant terminal condition $v_K(T,a) \equiv m$. But for $a \in \mathbb{R}$ fixed, $v_K(t,a)$ only depends on $T-t$, and since we intend to use a scaling argument it is convenient to turn to a forward setting. Then it suffices to show that the solution v_K to

$$v_K(t,a) \;=\; m \;-\; \Pi_a \int_0^t \mathrm{d}r\; (\varrho_\sigma \wedge K)(W_r)\, v_K^2(t-r, W_r) \tag{27}$$

is non-increasing in $t>0$ if $a=c=0$.

$2°$ (*scaling*) For $t>0$ fixed, introduce

$$u_t(s,a) := v_K\big(ts, \sqrt{t}\,a\big), \qquad s>0,\;\; a \in \mathbb{R}.$$

Then by Brownian scaling, from (27) we conclude

$$u_t(s,a) \;=\; m - \Pi_a \int_0^s \mathrm{d}r\; \varrho^t(W_r)\; u_t^2(s-r, W_r) \tag{28}$$

with

$$\varrho^t(b) := t(\varrho_\sigma \wedge K)\big(\sqrt{t}\,b\big) \;=\; \frac{\theta\, t^{1-\sigma/2}}{|b|^\sigma} \wedge (tK)\,, \qquad b \in \mathbb{R}. \tag{29}$$

These new bounded coefficients ϱ^t in the absorption term of (28) are non-decreasing in $t>0$ since $\sigma \le 2$. Hence the (unique) solutions u_t of equation (28) are non-increasing in $t>0$. In particular, $u_t(1,0) = v_K(t,0)$ is non-increasing in $t>0$. This finishes the proof. ■

Remark 12 (limitation to $\sigma \le 2$) This is the first time in the present development we needed to restrict to $\sigma \le 2$. ◇

3.6 Limiting function

Turning back to the truncated equation (23) or (25), we now replace the terminal function φ by $\varphi_K \in b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$, and assume that $\varphi_K \downarrow \varphi \in b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$ as $K \to \infty$. Then from monotonicity in both φ_K and $\varrho_\sigma \wedge K$, we obtain, for the corresponding solutions $v_K := V^K\varphi_K$, the following *pointwise limit assertion:*

$$v_K = V^K\varphi_K \downarrow \text{some } v =: V\varphi \ge 0 \;\text{ as }\; K \to \infty. \tag{30}$$

Note that at this stage the limiting $v = V\varphi$ could depend on the choice of the approximating sequence φ_K.

Lemma 13 (independence of the choice of the φ_K) *For φ in $b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$ fixed, the limiting function $v = V\varphi$ of (30) is independent of the choice of the approximating functions $\varphi_K \downarrow \varphi$.*

Proof Consider two sequences $\varphi_K \downarrow \varphi$ and $\psi_K \downarrow \varphi$. We may assume that $\varphi_K \leq \psi_K$ (otherwise bound φ_K and ψ_K between $\varphi_K \wedge \psi_K$ and $\varphi_K \vee \psi_K$). Then $V^K \varphi_K \leq V^K \psi_K$, and using the equation (23) or (25), and monotonicity, we may continue with

$$V^K \psi_K \leq V^K \varphi_K + \widetilde{\Pi}_{s,\omega_s} \Big[\psi_K(\widetilde{W}_T) - \varphi_K(\widetilde{W}_T) \Big].$$

Then the claim immediately follows by bounded convergence as $K \to \infty$. ■

Remark 14 (monotonicities) It is clear that the statements of the Lemmas 9 and 11 remain valid also for the limiting function $v = V\varphi$. ◇

Remark 15 (self-similarity) In the case $\sigma = 2$, $c = 0$, and for constant terminal functions m, the limiting functions $v = Vm$ are *self-similar* with respect to the Brownian scaling. In fact, for our approximating v_K but in a forward setting we have $v_K(L^2 t, La) = v_{L^2 K}(t,a)$, for each $L > 0$, since (27) is uniquely solvable. Letting $K \to \infty$ gives the claim. ◇

3.7 Disappearance at the catalytic center

The limiting functions $v = V\varphi$ vanish at the catalytic center c in the following sense, without any additional assumptions on φ.

Lemma 16 (disappearance at the catalytic center) *For φ in $b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$, the limiting function $v = V\varphi$ of* (30) *has the following property:*

$$v(s,\omega_s) = 0 \quad if \quad (\omega_s)_s = c, \ \omega_s \in \mathbf{C}^s, \ s \in [0,T).$$

Proof Set again $c = 0$. Since $V\varphi \geq 0$ is non-decreasing in φ, without loss of generality we may assume that φ equals a constant $m > 0$. Start by considering an approximating solution v_K of (23) with terminal condition m, for a fixed $K > 1$.

For any $[s,\omega_s] \in I \widehat{\times} \mathbf{C}^{\bullet}$, by Lemma 9 (and recalling Remark 10) we may write $v_K(s,\omega_s) = v_K\big(s, |(\omega_s)_s|\big)$. Additionally, by the monotonicity statement in that lemma, we may continue with $v_K(s,\omega_s) \geq v_K(s,0) \geq v(s,0)$. Applying this to the integral term in (23) we get in particular

$$0 \leq \widetilde{\Pi}_{0,0} \int_0^T \mathrm{d}r \ (\varrho_\sigma \wedge K)(W_r) \ v^2(r,0) \leq m.$$

Additionally, by monotone convergence as $K \to \infty$ we therefore conclude that

$$\Pi_0 \int_0^T \mathrm{d}r \ \varrho_\sigma(W_r) \ v^2(r,0) \leq m.$$

However,

$$\Pi_0\, \varrho_\sigma(W_r) = S_r\, \varrho_\sigma\,(0) \equiv +\infty, \qquad r > 0,$$

since $\sigma \geq 1$. Thus

$$v(r,0) = 0 \quad \text{for almost all} \quad r \in [0,T). \tag{31}$$

Finally, combined with the monotonicity statement in Lemma 11 (recall Remark 14)), we get $v(r,0) \equiv 0$ on $[0,T)$. Hence, the claim $v(s,\omega_s) = 0$ follows, finishing the proof. ■

3.8 Limiting equation

The *main result* of this section is the following proposition which states that the limiting function v introduced in (30) solves the formal limiting equation (as $K \to \infty$) arising from the Feynman-Kac equation (25), if we additionally switch to the Brownian path process $\widetilde{W}^c$ *killed* at c. Recall the supremum expression α_T of Lemma 6, and that $1 \leq \sigma < 2$.

Proposition 17 (limiting equation) *For $T > 0$ fixed and φ in $b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$, consider the limiting function $v = V\varphi \geq 0$ of* (30).

(a) (existence) *$v = V\varphi$ solves the Feynman-Kac equation*

$$v(s,\omega_s) \;=\; \widetilde{\Pi}^c_{s,\omega_s}\, \varphi(\widetilde{W}^c_T) \exp\Big[-\int_s^T \mathrm{d}r\;\, \varrho_\sigma(W^c_r)\, v\big(r, \widetilde{W}^c_r\big)\Big], \tag{32}$$

$[s,\omega_s] \in [0,T)\widehat{\times}\mathbf{C}^\bullet$.

(b) (uniqueness) *If $\sigma < 2$, or $\|\varphi\|_\infty\, \alpha_T < 1$, then $v = V\varphi$ is the unique element in $b\mathcal{B}\big[I\widehat{\times}\mathbf{C}^\bullet, \mathbb{R}_+\big]$ which solves* (32).

(c) (continuity) *If $\sigma < 2$, or $\|\varphi\|_\infty\, \alpha_T < 1$, then $v = V\varphi$ continuously depends on φ in the topology of uniform convergence.*

Recall that the symbol $\widetilde{W}^c_T$ in (32) tacitly requires that $\widetilde{W}^c$ is still alive at time T. Note also that (32) can symbolically be written as

$$-\tfrac{\partial}{\partial s}\, v(s,\omega_s) \;=\; \widetilde{A}_s v(s,\omega_s) - \varrho_\sigma\big((\omega_s)_s\big)\, v^2(s,\omega_s), \tag{33}$$

with terminal condition $v(T,\cdot\,) = \varphi$, and boundary condition $v(s,\omega_s)\big|_{(\omega_s)_s = c} = 0,\;\; s < T$. We also mention that there are versions of the uniqueness and continuity statements (b) and (c) *without* the additional assumptions; see Theorem 19 (c) and (b) at p.23.

Proof of Proposition 17

1° (*existence*) We want to show that the limiting function $v = V\varphi$ solves equation (32). To this aim, fix $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$, and $[s,\omega_s] \in [0,T)\widehat{\times}\mathbf{C}^\bullet$. If

$(\omega_s)_s = c$ holds, then the r.h.s. of (32) disappears by the property (21) of the subprobabilities $\widetilde{\Pi}^c$. But by Lemma 16 also the l.h.s. vanishes. Therefore we may restrict our attention to the case $(\omega_s)_s \neq c$.

Going back to the approximating functions $v_K = V_K\varphi$, we look at the truncated Feynman-Kac equation (25). First we restrict the expectation at the r.h.s. (of (25)) to the event $\{\tau^c \leq T\}$ (with τ^c the hitting time of the catalytic center) and want to show that this results in a negligible term as $K \to \infty$. In fact, this part of the expectation can be bounded from above by restricting the integral in the exponent to $r > \tau^c$. Next we use the strong Markov property at time τ^c, and the uniqueness of the solutions to (25). Then the resulting upper bound of this part of the r.h.s. of (25) can be written as

$$\widetilde{\Pi}_{s,\omega_s} \mathbf{1}\{\tau^c \leq T\}\, v_K\big(\tau^c, \widetilde{W}_{\tau^c}\big).$$

By monotone convergence this tends to $\widetilde{\Pi}_{s,\omega_s} \mathbf{1}\{\tau^c \leq T\}\, v\big(\tau^c, \widetilde{W}_{\tau^c}\big)$ as $K \to \infty$. However, this vanishes, since by $(\widetilde{W}_{\tau^c})_{\tau^c} = W_{\tau^c} = c$ the latter v-expression disappears on the event $\{\tau^c < T\}$ according to Lemma 16, and since τ^c has a continuous law.

It remains to show that

$$\widetilde{\Pi}_{s,\omega_s} \mathbf{1}\{\tau^c > T\}\, \varphi(\widetilde{W}_T) \exp\Big[-\int_s^T \mathrm{d}r\ (\varrho_\sigma \wedge K)(W_r)\, v_K\big(r, \widetilde{W}_r\big)\Big] \tag{34}$$

converges as $K \to \infty$ to the analogous expression without involving the K (recall the identity (21)). This will be provided via two-sided estimates.

First of all, to estimate from above, switch in (34) from v_K to v, and let K tend to ∞ after this. Then the desired limit term will come out by monotone convergence based on $(\varrho_\sigma \wedge K) \uparrow \varrho_\sigma$.

Concerning the other direction, pass from $(\varrho_\sigma \wedge K)$ to ϱ_σ in (34). If we assume for the moment that a.s. we still have a finite integral in the exponent, than again by monotone convergence we will be done. To demonstrate the above-mentioned finiteness, it suffices to show that the (weighted) expectation of the new integral in the exponent is finite:

$$\widetilde{\Pi}_{s,\omega_s} \mathbf{1}\{\tau^c > T\}\, \varphi(\widetilde{W}_T) \int_s^T \mathrm{d}r\ \varrho_\sigma(W_r)\, v_K\big(r, \widetilde{W}_r\big) < +\infty.$$

But from (25),

$$0 \leq v_K(r,\cdot) \leq \widetilde{\Pi}_{r,\cdot}\, \varphi(\widetilde{W}_T) \leq \|\varphi\|_\infty\,,$$

and we get the bound

$$\widetilde{\Pi}_{s,\omega_s} \mathbf{1}\{\tau^c > T\} \int_s^T \mathrm{d}r\ \varrho_\sigma(W_r)$$

which is finite by Lemma 6 and the time-homogeneity of Brownian motion.

2° (*uniqueness*) Assume that v^1 and v^2 are *different* solutions of (32). Let $s_0 \leq T$ denote the supremum over all $s < T$ such that $v^1(s,\cdot) \neq v^2(s,\cdot)$. Fix for the moment $0 \leq s < s_0$ and $\omega_s \in \mathbf{C}^s$. Next we search for an upper bound for $|v^1(s,\omega_s) - v^2(s,\omega_s)|$ by using the equation (32). To this aim, split up the common part of the integral in the exponent, estimate the related exponential term by 1, and use the elementary inequality $|\mathrm{e}^{-x} - \mathrm{e}^{-y}| \leq |x-y|$, $x,y \geq 0$, to get the bound

$$\widetilde{\Pi}^c_{s,\omega_s}\, \varphi(\widetilde{W}^c_T) \int_s^{s_0} \mathrm{d}r\ \varrho_\sigma(W^c_r) \left| v^1\bigl(r,\widetilde{W}^c_r\bigr) - v^2\bigl(r,\widetilde{W}^c_r\bigr)\right|.$$

Denoting (in this proof) by $\|\cdot\|$ the supremum norm on $[0,s_0)\widehat{\times}\mathbf{C}^\bullet$, and using the time-homogeneity of the Brownian motion killed at c, we conclude that

$$\|v^1 - v^2\| \;\leq\; \|\varphi\|_\infty \|v^1 - v^2\| \sup_a \Pi_a \mathbf{1}\{\tau^c > s_0 - s\} \int_0^{s_0-s} \mathrm{d}r\ \varrho_\sigma(W_r).$$

The supremum expression is finite and equals $\alpha_{s_0-s} = (s_0 - s)^{1-\sigma/2}\alpha_1$, by Lemma 6. If $\sigma = 2$, we get a contradiction by our assumption, otherwise a contradiction follows by choosing s sufficiently close to s_0.

3° (*continuity*) Use the same arguments as in step 2°, where for the case $\sigma < 2$ the interval $[0,T]$ has additionally be decomposed equidistantly in N pieces with choosing N satisfying $\|\varphi\|_\infty\, \alpha_{T/N} < 1$, and proceed by induction on N. (Later we will use the continuity claim only for "small" φ.) This finishes the proof. ■

4 Historical hyperbolic super-Brownian motion

The purpose of this section is the construction of the hyperbolic SBM X in $\mathbb{R}$ as claimed in the existence Theorem 3. Actually we will construct the related *historical* superprocess $\widetilde{X}$. This can be done by starting from the historical SBM $\widetilde{X}^K$ with truncated branching rate $\varrho_\sigma \wedge K$ and passing to the limit as $K \to \infty$.

4.1 Semigroup structure of limiting functions

In the previous section, all paths w ended at time T. Now we write t instead of T and think of t as a variable. To again have a unified reference space, we replace $\mathcal{C}\bigl[[0,t],\mathbb{R}\bigr]$ by $\mathcal{C}[\mathbb{R}_+,\mathbb{R}] =: \mathbf{C}$ endowed with the topology of uniform convergence on bounded intervals (Polish space). $\mathbf{C}^s$ is again the closed subspace of all continuous paths stopped at time s. Also the other notations of the previous section are modified in the obvious way.

Take $\varphi \in b\mathcal{B}[\mathbf{C},\mathbb{R}_+]$. Fix $t \geq 0$ for the moment. Recall the solution $V^K\varphi$ of the truncated log-Laplace equation (23) with T replaced by t, and similarly $V\varphi$ for its limit as $K \uparrow \infty$. We write now more carefully

$$V_{s,t}\,\varphi\,(\omega_s) := V\varphi\,(s,\omega_s), \qquad 0 \leq s \leq t, \quad \omega_s \in \mathbf{C}^s, \tag{35}$$

to exhibit the dependence on t, and define $V^K_{s,t}\varphi(\omega_s)$ analogously based on $V^K\varphi(s,\omega_s)$.

Lemma 18 (semigroup structure) *The limiting functions of* (30) (*using the notation of* (35)) *satisfy*

$$V_{s,r}V_{r,t}\,\varphi = V_{s,t}\,\varphi, \qquad 0 \le s \le r \le t, \quad \varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+].$$

Proof Fix s, r, t, φ as in the lemma. From equation (23) or (25) we get

$$V^K_{s,r}V^K_{r,t}\,\varphi = V^K_{s,t}\,\varphi, \qquad K > 1, \quad 0 \le s \le r \le t, \quad \varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+].$$

Then the claim follows from the limit assertion (30) and the independence of choice Lemma 13. ■

4.2 Existence of the historical hyperbolic SBM $\widetilde{X}$

For $K > 1$, let

$$\widetilde{X}^K = \Big[\widetilde{X}^K,\, \widetilde{P}^K_{s,\mu},\, s \ge 0,\, \mu \in \mathcal{M}[\mathbf{C}^s]\Big]$$

denote the *historical SBM* related to the *truncated* branching rate $\varrho_\sigma \wedge K$, that is, a (time-inhomogeneous) strong Markov process with states $\widetilde{X}^K_t \in \mathcal{M}[\mathbf{C}^t]$, $t \ge s$, having the following Laplace transition functional

$$\widetilde{P}^K_{s,\mu} \exp\big\langle \widetilde{X}^K_t, -\varphi\big\rangle = \exp\big\langle \mu, -V^K_{s,t}\,\varphi\big\rangle, \tag{36}$$

$0 \le s \le t$, $\mu \in \mathcal{M}[\mathbf{C}^s]$, $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$. Here $V^K_{\cdot,t}\varphi \ge 0$ uniquely solves the truncated equation (23) (with T replaced by t). For a detailed exposition we refer e.g. to Dawson and Perkins [DP91, Chapter 2] or to Dynkin [Dyn91b]; see also Mueller and Perkins [MP92].

The *interpretation* is that $\widetilde{X}^K_t(\mathrm{d}w)$ describes the mass of all particles at time t with location w_t but only those which (or whose ancestors) moved during the time interval $[s,t]$ along the curve $\{w_r;\ s \le r \le t\}$. In this sense, $\widetilde{X}^K$ is a refinement of the usual continuous super-Brownian motion X^K with truncated branching rate $\varrho_\sigma \wedge K$.

Now we are ready to introduce and characterize our limiting process.

Theorem 19 (historical hyperbolic SBM $\widetilde{X}$)

(a) (convergence) *There is a (time-inhomogeneous) Markov process* $\widetilde{X} = \big[\widetilde{X},\, \widetilde{P}_{s,\mu},\, s \ge 0,\, \mu \in \mathcal{M}[\mathbf{C}^s]\big]$ *with states* $\widetilde{X}_t \in \mathcal{M}[\mathbf{C}^t]$, $t \ge s$, *such that the historical SBM* $\widetilde{X}^K$ *with truncated branching rate* $\varrho_\sigma \wedge K$ *(and non-killed Brownian motion* W *as motion law) tends to* $\widetilde{X}$ *as* $K \to \infty$ *in the sense of convergence of all finite-dimensional distributions.*

(b) (Laplace transition functional) $\widetilde{X}$ *is characterized by the following Laplace transition functional:*

$$\widetilde{P}_{s,\mu} \exp \langle \widetilde{X}_t \, , -\varphi \rangle \; = \; \exp \langle \mu, -V_{s,t}\, \varphi \rangle, \tag{37}$$

$0 \le s \le t$, $\mu \in \mathcal{M}[\mathbf{C}^s]$, $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$. *Here* $V_{\cdot,t}\, \varphi \ge 0$ *is the limiting function of* (30) *at p.18* (*with* T *replaced by* t, *and in the writing as introduced in* (35)).

(c) (log-Laplace equation) $v = V_{\cdot,t}\, \varphi$ *is the unique analytic element in* $b\mathcal{B}\big[I \widehat{\times} \mathbf{C}^\bullet, \mathbb{R}_+\big]$ *(that is the maps* $\lambda \mapsto V_{s,t}\,(\lambda\varphi)(\omega_s)$, $\lambda \ge 0$, *are analytic), which solves the Feynman-Kac equation*

$$v(s, \omega_s) = \widetilde{\Pi}^c_{s,\omega_s}\, \varphi(\widetilde{W}^c_t) \exp\Big[- \int_s^t \mathrm{d}r \;\, \varrho_\sigma(W^c_r)\, v\big(r, \widetilde{W}^c_r\big) \Big], \tag{38}$$

$[s, \omega_s] \in [0,t) \widehat{\times} \mathbf{C}^\bullet$.

(d) (moments) *The following expectation and variance formulas hold:*

$$\begin{aligned} \widetilde{P}_{s,\mu} \langle \widetilde{X}_t \, , \varphi \rangle \; &= \; \widetilde{\Pi}^c_{s,\mu}\, \varphi\big(\widetilde{W}^c_t\big), \\ \widetilde{\mathrm{V}}\mathrm{ar}_{s,\mu} \langle \widetilde{X}_t \, , \varphi \rangle \; &= \; 2\, \widetilde{\Pi}^c_{s,\mu} \int_s^t \mathrm{d}r \;\, \varrho_\sigma(W^c_r) \Big[\widetilde{\Pi}^c_{r, \widetilde{W}^c_r}\, \varphi\big(\widetilde{W}^c_t\big) \Big]^2, \end{aligned}$$

$0 \le s < t$, $\mu \in \mathcal{M}[\mathbf{C}^s]$, $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$.

$\widetilde{X}$ is called the *historical hyperbolic super-Brownian motion* in $\mathbb{R}$. Note that $\widetilde{X}_t$ is a measure on continuous paths w on $\mathbb{R}_+$ stopped at time t. It describes the ancestry of all particles alive at time t.

As a rough interpretation of the expectation formula in (d) one could say: The “expectation” of the historical superprocess $\widetilde{X}$ is given by the Brownian path process $\widetilde{W}^c$ with killing at c.

Remark 20 (supercritical σ) If we start the construction with Brownian motion killed at c (rather than with the non-killed one), the range of the exponent σ in the branching rate ϱ_σ can be extended to $\sigma < 3$. $\diamond$

Before we turn to the proof of this theorem in the next subsection, we want to show how it implies the *existence Theorem* 3 of p.4.

Proof of Theorem 3 By the *projection*

$$X_t(B) := \widetilde{X}_t\Big(\big\{ w \in \mathbf{C}^t;\; w_t \in B \big\} \Big), \qquad \text{Borel } B \subseteq \mathbb{R}, \tag{39}$$

we define the *hyperbolic super-Brownian motion* $X = \big[X, P_\mu\, , \, \mu \in \mathcal{M}\big]$ in $\mathbb{R}$ (statements (a) and (c) of Theorem 3, except the claimed non-degeneration). Note that X (as opposed to $\widetilde{X}$) is a time-homogeneous $\mathcal{M}$-valued Markov process.

From the expectation formula in (d) it follows

$$P_\mu \langle X_t\,,\varphi\rangle \;=\; \int\!\mu(\mathrm{d}a)\; S_t^c\varphi\,(a), \qquad \mu\in\mathcal{M}, \quad t>0, \quad \varphi\in b\mathcal{B}_+\,, \tag{40}$$

in particular,

$$E\big\{X_t(\mathsf{IR})\big|\,X_0\big\} \;=\; \int\!X_0(\mathrm{d}a)\;\Pi_a(\tau^c>t)\;<\;X_0(\mathsf{IR}), \tag{41}$$

$X_0\neq 0,\ t>0$. Hence, the *total mass process* $t\mapsto X_t(\mathsf{IR})$ is a *supermartingale* but no longer a martingale (part (b) of Theorem 3), as opposed to the critical superprocesses with locally finite catalytic mass. In fact, since the underlying Brownian motion is killed at the center c of the catalytic medium, no mass can be born at c and the expectation of the process is not preserved (except the zero mass), despite the otherwise criticality of the branching mechanism.

The variance formula in (d) specializes for X as follows:

$$\mathrm{Var}_\mu \langle X_t\,,\varphi\rangle \;=\; 2\int_0^t\!\mathrm{d}r \int_{a\neq c}\!\mu(\mathrm{d}a)\,\int\!\mathrm{d}b\;\varrho_\sigma(b)\,\mathrm{p}_r^c(a,b)\,[S_{t-r}^c\varphi]^2(b), \tag{42}$$

$\mu\in\mathcal{M},\ \varphi\in b\mathcal{B}_+\,,\ t>0$. In particular,

$$\mathrm{Var}_{\delta_a} X_t(\mathsf{IR}) \;=\; 2\int_0^t\!\mathrm{d}r\int\!\mathrm{d}b\;\varrho_\sigma(b)\,\mathrm{p}_r^c(a,b)>0, \qquad a\neq c, \quad t>0. \tag{43}$$

Note that the latter expression is *finite*, provided that $\sigma<2$ (moderate case), whereas it is *infinite* for $\sigma=2$ (critical hyperbolic branching rate). In fact, only the influence of the singularity for $b\to c$ of the branching rate ϱ_σ has to be checked, for $a\neq c$. But for this we can apply the bounds in Lemma 5. (Consequently, the total mass process $X(\mathsf{IR})$ has finite variance if and only if the branching functional $K(\mathrm{d}r)=\varrho_\sigma(W_r^c)\,\mathrm{d}r$ has finite characteristic, as noticed at p.5.)

Since the variance (43) is not zero, the hyperbolic super-Brownian motion X is *non-degenerate*. This completes the proof of Theorem 3. ■

Remark 21 (total mass convergence) It can be expected that there is a continuous modification of the total mass process $t\mapsto X_t(\mathsf{IR})$. Hence, as a continuous non-negative supermartingale it converges almost surely to 0 as $t\to\infty$ since the expectation will vanish, recall (41). ◇

Remark 22 (stochastic equation) It can also be expected that the hyperbolic super-Brownian motion X lives on the set of absolutely continuous measures, and that there is a density field jointly continuous on $\{t>0\}\times\mathsf{IR}$ satisfying the stochastic equation (1). Setting formally $\varphi=\delta_c$ in (40) and (42) suggests that this density field vanishes identically at the catalytic center (as opposed to the single point-catalytic super-Brownian motion [DF94] where the variance of the density field blows up approaching the catalyst and the density field is non-zero at the catalyst's position at some random times). ◇

4.3 Proof of the existence theorem

Now we provide the *Proof of Theorem* 19. Fix $0 \leq s < t$, $\mu \in \mathcal{M}[\mathbf{C}^s]$, and $\varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+]$. Passing in (36) to the monotone limit $V^K_{s,t}\,\varphi \downarrow V_{s,t}\,\varphi$ as $K \uparrow \infty$, the Laplace functionals

$$L^K_{s,t,\mu}(\varphi) := \widetilde{P}^K_{s,\mu} \exp\left\langle \widetilde{X}^K_t, -\varphi \right\rangle \tag{44}$$

have a limit functional $\exp\left\langle \mu, -V_{s,t}\,\varphi \right\rangle := L^\infty_{s,t,\mu}(\varphi)$. By Proposition 17 (c), the limit $L^\infty_{s,t,\mu}(\varphi)$ continuously depends on φ provided that φ is sufficiently "small" (even in the topology of uniform convergence). Hence, there is a law $Q_{s,t,\mu}$ of a random measure in $\mathcal{M}[\mathbf{C}^t]$ which Laplace functional, denoted by $L_{s,t,\mu}(\varphi)$, coincides with $L^\infty_{s,t,\mu}(\varphi)$ for small φ, and

$$L^K_{s,t,\mu}(\varphi) \longrightarrow L_{s,t,\mu}(\varphi) \quad \text{as} \quad K \to \infty \quad \text{for all} \quad \varphi \in b\mathcal{B}[\mathbf{C}, \mathbb{R}_+];$$

see Leduc [Led95, Lemma 4.10] (cf. also Dynkin [Dyn94, 3.3.4.C]). Therefore $L^\infty_{s,t,\mu} = L_{s,t,\mu}$. Since by Lemma 18 the operators $V_{s,t}$ form a semigroup, the laws $Q_{s,t,\mu}$ form a family of transition probabilities of a Markov process, the desired historical hyperbolic SBM $\widetilde{X}$.

Consequently, we proved the existence of $\widetilde{X}$, the convergence claim (a), and the representation of Laplace transitions functionals (b). The analyticity of $\lambda \mapsto V_{s,t}\,(\lambda\varphi)(\omega_s)$, follows from the analyticity of the Laplace function $\lambda \mapsto L_{s,t,\mu}(\lambda\varphi)$ where $\mu = \delta_{\omega_s}$. Uniqueness of $V_{\cdot,t}(\lambda\varphi)$ for small λ was established in Proposition 17 (b), and by analytic continuation we get the uniqueness in the case $\lambda = 1$. Based on the representation (37) of Laplace transition functionals, by standard arguments, the above-mentioned moment formulas in (d) can be derived. This finishes the proof. ■

4.4 Stopped historical hyperbolic super-Brownian motion

There is a further refinement of the historical super-Brownian motion $\widetilde{X}^K$ with truncated branching rate $\varrho_\sigma \wedge K$, which goes back to Dynkin [Dyn91a, § 1.5]. In fact, for $s \geq 0$, let $\mathcal{T}_s$ denote the set of all *finite s–stopping times* τ with respect to the (natural) filtration of Brownian path process $\widetilde{W}$. Then there is a family

$$\left\{\widetilde{X}^K_\tau;\ \tau \in \mathcal{T}_s\,,\ s \geq 0\right\} \tag{45}$$

called the *"stopped" historical super-Brownian motion* with truncated branching rate $\varrho_\sigma \wedge K$. The principal idea here is that $\widetilde{X}^K_\tau$ is a random measure on paths w stopped at time τ instead of t. These paths give the history of particles up to time τ, where τ may be different for each particle.

This family (45) satisfies the so-called *special Markov property,* which roughly says that at any $\tau \in \mathcal{T}_s$, the stopped historical SBM starts anew (see Dynkin [Dyn91a, Theorem 1.6]).

As in Theorem 19 (d), we have the following expectation and variance formulas:

$$\widetilde{P}^K_{s,\mu}\langle \widetilde{X}^K_\tau, \Phi\rangle = \widetilde{\Pi}_{s,\mu}\Phi(\widetilde{W}_\tau), \tag{46}$$

$$\widetilde{\mathrm{V}}\mathrm{ar}^K_{s,\mu}\langle \widetilde{X}^K_\tau, \Phi\rangle = 2\,\widetilde{\Pi}_{s,\mu}\int_s^\tau \mathrm{d}r\,\big(\varrho_\sigma \wedge K\big)(W_r)\Big[\Pi_{r,\widetilde{W}_r}\,\Phi\big(\widetilde{W}_\tau\big)\Big]^2, \tag{47}$$

$s \geq 0$, $\tau \in \mathcal{T}_s$, $\Phi \in b\mathcal{B}\big[\mathbf{C}, \mathbb{R}_+\big]$, and $\mu \in \mathcal{M}(\mathbf{C}^s)$; see e.g. Dynkin [Dyn91a, (1.50a)].

As with the existence Theorem 19, there is also a refinement of the historical hyperbolic SBM $\widetilde{X}$ as described above in the truncated case. After we already providing a rather detailed construction of the $\widetilde{X}$–process, at this point we only give a brief treatment. For $s \geq 0$, let $\mathcal{T}^c_s$ denote the set of all *finite s–stopping times* τ of Brownian path process $\widetilde{W}^c$ killed at c, strictly smaller than the "life time" of $\widetilde{W}^c$, actually the time of death of $\widetilde{W}^c$, which we also denote by τ^c. Then there is a family

$$\big\{\widetilde{X}_\tau\,;\ \tau \in \mathcal{T}^c_s,\ \ s \geq 0\big\}$$

called the *stopped historical hyperbolic super-Brownian motion.* Of course, $\widetilde{X}_\tau$ is again a random measure on paths stopped at time $\tau < \tau^c$, illustrating the historical population picture of all particles alive at the random moment τ.

This family also satisfies the *special Markov property,* saying that at any $\tau \in \mathcal{T}^c_s$, the stopped historical hyperbolic SBM starts anew.

The *expectation* formula (46) again reads as

$$\widetilde{P}_{s,\mu}\langle \widetilde{X}_\tau, \Phi\rangle = \widetilde{\Pi}^c_{s,\mu}\Phi\big(\widetilde{W}^c_\tau\big) = \widetilde{\Pi}_{s,\mu}\Phi\big(\widetilde{W}_\tau\big), \tag{48}$$

$s \geq 0$, $\tau \in \mathcal{T}^c_s$, $\Phi \in b\mathcal{B}\big[\mathbf{C}, \mathbb{R}_+\big]$, where we assume that the measure $\mu \in \mathcal{M}(\mathbf{C}^s)$ satisfies $\mu\big\{w \in \mathbf{C}^s;\ w_s = c\big\} = 0$ (note that this is no real restriction, since the particle's motion process is Brownian motion killed at c).

5 Killing around the critical hyperbolic pole

In this section we want to show that for the super-Brownian motion X in the critical hyperbolic medium ϱ_2 no mass will ever reach the catalytic center. That is, the particles hidden in the clouds die already *before* they "hit" c, that is before they are killed at c. This will be based on some methods involving historical superprocesses developed in Chapter 8 of Dawson and Perkins [DP91] to estimate the modulus of continuity of the support of superprocesses (see also Mueller and Perkins [MP92]).

5.1 Reformulation of the strong killing theorem

Actually, we restate Theorem 4 of p.5 at the level of historical superprocesses in the following way.

Theorem 23 (strong killing in the case of the critical ϱ_2) *Fix $a \neq c$. Under $\sigma = 2$, there exists an increasing sequence of stopping times τ_n of Brownian path process $\widetilde{W}^c$ killed at c, strictly smaller than the life time τ^c of $\widetilde{W}^c$, such that*

$$\widetilde{P}_{0,\delta_a}\Big(\widetilde{X}_{\tau_n} = 0 \ \ except\ for\ finitely\ many\ \ n\Big) = 1.$$

Remark 24 (moderate hyperbolic case) As opposed to the killing Theorem 4, in the *moderate* hyperbolic case $1 \leq \sigma < 2$ the population mass *can* reach the catalytic center c. In fact, putting $c = 0$, by an adoption of Dynkin's Theorem 8.1, p.1239 in [Dyn93], to the present situation, starting with $X_0 = \delta_a$, $a > 0$, the probability that the range of X does not hit c is given by $\mathrm{e}^{-\overline{v}(a)}$ where $\overline{v}$ is the maximal solution to $\frac{1}{2}\Delta v = \varrho_\sigma v^2$ on $(0, +\infty)$. But setting $p := 2 - \sigma$, the function $v(a) := \frac{1}{2\theta} p(p+1)\, a^{-p},\ a > 0$, is a strictly positive solution of that equation. Therefore the above probability is (strictly) *less* than 1. ◇

For the proof of Theorem 23, without loss of generality we may set $c = 0$ and $a > 0$. To simplify notation, we identify δ_a with a and write $\widetilde{P}_a$ instead of $\widetilde{P}_{0,\delta_a}$. By this choice of the initial state, we may restrict to particle paths w in

$$\mathbf{C}_a := \Big\{ w \in \mathbf{C} = \mathcal{C}[\mathbb{R}_+, \mathbb{R}];\ w_0 = a > 0 \Big\},$$

ignoring that particles are killed if they reach $c = 0$. This is justified since actually we will observe these paths at most until they reach the catalytic center $c = 0$. So this change in the space of particles' paths has no effect. More precisely, for $w \in \mathbf{C}_a$, we pay attention to the following increasing sequence of *hitting times*

$$\tau_n := \tau(2^{-n}a), \qquad n \geq 0, \tag{49}$$

where $\tau(b) := \tau^b$ denotes the (first) time the path w hits the point $b \in \mathbb{R}$. (Recall that by the recurrence of one-dimensional Brownian motion each hitting time $\tau(b)$ is finite Π_a–a.s.) In § 5.2 we will actually use this monotone sequence $\tau_0, \tau_1, \ldots$ of hitting times smaller than τ^0, the life time of killed Brownian path process, to prove Theorem 23. For this purpose it *suffices* to show that

$$\widetilde{P}_a\big(\widetilde{X}_{\tau_n} \neq 0\big) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{50}$$

In fact, then these probabilities are summable along a subsequence, and from Borel-Cantelli we conclude for the existence of a smallest (random) integer N (concerning this subsequence) such that $\widetilde{X}_{\tau_N} = 0$ with $\widetilde{P}_a$–probability one, then proving Theorem 23.

Remark 25 Note that the random integer N depends on more than just a single Brownian path. Therefore τ_N is *not* a stopping time of killed Brownian motion, and X_{τ_N} does not belong to Dynkin's family of stopped measures. ◇

As a technical preparation for the proof of (50), in this subsection we still state the following simple property of the Brownian motion, which equivalently could be reformulated in terms of the Brownian motion W^c killed at c, or even for $\widetilde{W}^c$.

Lemma 26 (infinite accumulated branching rate) *Along a Brownian path W until it reaches 0, the accumulated rate of branching is infinite:*

$$\int_0^{\tau^0} \mathrm{d}t\ \varrho_2(W_t) = \infty \quad \Pi_a\text{–}a.s., \qquad a > 0.$$

Proof First of all, using the hitting times τ_n from (49),

$$\int_0^{\tau^0} \mathrm{d}t\ \varrho_2(W_t)\ \geq\ \sum_{n=0}^{\infty} Q_n\,,$$

where

$$Q_n := \int_{\tau_n}^{\tau_{n+1}} \mathrm{d}t\ \mathbf{1}\Big\{W_t \leq 2^{-n}a\Big\}\,\varrho_2(2^{-n}a), \qquad n \geq 0. \tag{51}$$

Since $Q_0 > 0$ with Π_a–probability one, it suffices to show that the $Q_0\,, Q_1\,, \ldots$ are independent and identically distributed (with respect to Π_a).

The independence immediately follows from the strong Markov property. We want to calculate $\Pi_a\big(Q_n \leq r\big)$, $r > 0$. Again by the strong Markov property but also time- and space-homogeneity as well as symmetry, it equals

$$\Pi_0\bigg(\theta 2^{2n}a^{-2}\int_0^{\tau(2^{-n-1}a)} \mathrm{d}t\ \mathbf{1}\big\{W_t \geq 0\big\}\ \leq\ r\bigg).$$

Now we can use the self-similarity of standard Brownian motion starting from the origin to continue with

$$=\ \Pi_0\bigg(\theta a^{-2}\int_0^{\tau(a/2)} \mathrm{d}t\ \mathbf{1}\big\{W_t \geq 0\big\} \leq r\bigg)\ =\ \Pi_a\big(Q_0 \leq r\big)$$

finishing the proof. ∎

Now we restrict our attention to paths $w \in \mathbf{C}_a$ with infinite accumulated rate of branching as in Lemma 26.

5.2 Proof of the strong killing theorem

Recall that for the proof of Theorem 23 it suffices to show the convergence statement (50).

From Lemma 26 we conclude for the existence of positive sequences $\varepsilon_n \to 0$ and $\xi_n \to \infty$ as $n \to \infty$ such that

$$\Pi_a(E_n) \geq 1 - \varepsilon_n^2 \quad \text{where} \quad E_n := \Big\{w \in \mathbf{C}_a\,;\ \int_0^{\tau_n} \mathrm{d}t\ \varrho_2(w_t) \geq \xi_n\Big\}. \tag{52}$$

Note that E_n belongs to $\mathcal{F}_{\tau_n}$, the σ–field generated by w_t for $t \le \tau_n$.

Here is the *intuitive picture* for the further procedure. For a fixed large n, according to (52) there is only a small Π_a-chance that a Brownian path belongs to the complement E_n^c of E_n (in $\mathbf{C}_a$). Therefore, this set E_n^c of *"bad paths"* has a small stopped historical $\widetilde{X}_{\tau_n}$-measure, with a high $\widetilde{P}_a$–probability. Then using Iscoe's [Isc88] techniques, we will show that the set of related particles is likely to die out before time τ_{n+1}. Also, for the original set E_n of *good paths* we will show that with high probability, the related particles have died out before time τ_n, by the huge accumulated rate of branching. This will be the more difficult part of the proof and will be provided by some time change argument and finally by a comparison with Feller's branching diffusion.

Now we give the details along these lines to arrive at the claim (50).

Step 1° We set

$$\lambda_n := \widetilde{X}_{\tau_n}(E_n^c). \tag{53}$$

By the expectation formula (48) with $s = 0$, $\tau = \tau_n$ and $\Phi = \mathbf{1}\{E_n^c\}$, we have

$$\widetilde{P}_a\,\lambda_n = \widetilde{P}_a\,\widetilde{X}_{\tau_n}(E_n^c) \,=\, \widetilde{\Pi}_{0,a}\big(\widetilde{W}_{\tau_n} \notin E_n\big) \le \Pi_a(E_n^c) \,\le\, \varepsilon_n^2\,.$$

Using Markov's inequality, we therefore conclude

$$\widetilde{P}_a\big(\lambda_n \ge \varepsilon_n\big) \,\le\, \varepsilon_n \,\to\, 0 \quad \text{as} \quad n \to \infty. \tag{54}$$

Consequently, the set E_n^c has small $\widetilde{X}_{\tau_n}$-measure λ_n with high $\widetilde{P}_a$–probability, as desired.

Step 2° Let us examine the further fate of the mass λ_n defined in (53) we need to study in the case $\lambda_n \to 0$. Fix n. By the definition of τ_n, at the moment τ_n the (projected) population X_{τ_n} will be concentrated at the space point $2^{-n}a$. Based on the special Markov property, we can start X anew after the time τ_n, namely with the mass λ_n attached to the point $2^{-n}a$.

Now we adapt Iscoe's [Isc88] analysis in the constant branching rate case to our situation of a critical hyperbolic medium ϱ_2. We estimate the probability that, starting with the mass λ_n, completely concentrated at the point $2^{-n}a$, the *range* of the arising superprocess will be contained in the surrounding space interval $I_n := (2^{-n-1}a,\, 3\cdot 2^{-n-1}a)$ (at which the branching rate ϱ_2 is bounded since a and n are fixed). This (conditional) probability, denoted by p_n, is given by

$$p_n \,=\, \exp\big[-\lambda_n\, u(2^{-n}a)\big] \tag{55}$$

where u satisfies

$$\tfrac{1}{2}\,\Delta u \,=\, \varrho_2\, u^2 \quad \text{on} \quad I_n\,, \qquad u\big|_{\partial I_n} = \infty,$$

(cf. Dynkin [Dyn93, Corollary II.8.1]). Since on I_n the critical hyperbolic branching rate ϱ_2 is not smaller than $\theta\, 3^{-2}2^{2n+2}a^{-2}$, we conclude that $u \le \overline{u}$, where $\overline{u}$ solves

$$\tfrac{1}{2}\,\Delta\overline{u} \;=\; \theta\,3^{-2}2^{2n+2}a^{-2}\,\overline{u}^{\,2} \quad \text{on} \quad I_n\,, \qquad \overline{u}\big|_{\partial I_n} = \infty,$$

(recall that $a > 0$ is the fixed starting point, and $\theta > 0$ is an additional weight of the branching rate ϱ_2). By scaling, we find that $v(b) := \overline{u}(2^{-n}b),\ b \in I_0\,,$ satisfies

$$\tfrac{1}{2}\,\Delta v \;=\; \theta\,3^{-2}2^{2}a^{-2}\,v^{2} \quad \text{on} \quad I_0 = \left(\tfrac{a}{2}, \tfrac{3a}{2}\right), \qquad v\big|_{\partial I_0} = \infty. \tag{56}$$

But then $u(2^{-n}a) \le \overline{u}(2^{-n}a) = v(a)$, which by uniqueness of the (maximal) solution to equation (56) does not depend on n. So finally we get

$$p_n \;\ge\; \exp\!\big[-\lambda_n\,v(a)\big] \tag{57}$$

converging to 1 if $\lambda_n \to 0$. Consequently, the small starting mass λ_n at time τ_n concentrated at $2^{-n}a$ will essentially not hit $2^{-n-1}a$ in the further development.

Step 3° Assume for the moment that

$$\widetilde{P}_a\Big(\widetilde{X}_{\tau_n}(E_n) > 0\Big) \;\longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{58}$$

Then we would have all ingredients needed to show (50). Indeed, recalling the definition (53) of λ_n, by the special Markov property we have

$$\left.\begin{aligned}\widetilde{P}_a\Big(\widetilde{X}_{\tau_{n+1}} \ne 0\Big) \;\;\le\;\; & \widetilde{P}_a\Big(\widetilde{X}_{\tau_n}(E_n) > 0\Big) + \widetilde{P}_a(\lambda_n \ge \varepsilon_n)\\ & + \widetilde{P}_a\Big\{P_{\tau_n,\lambda_n\delta_{2^{-n}a}}\big(X_{\tau_{n+1}} \ne 0\big)\,\Big|\,\lambda_n < \varepsilon_n\Big\}.\end{aligned}\right\} \tag{59}$$

In fact, in order that the historical process $\widetilde{X}$ is not extinct at time τ_{n+1}, either at time τ_n we have some particles with path in E_n, or we do not have such particles. In the latter case, we must have particles with a path in E_n^c. But then their mass λ_n is either larger than ε_n or smaller. If their mass is smaller than ε_n, we use the fact that the (projected) superprocess X starts anew at time τ_n with this mass λ_n concentrated at $2^{-n}a$ and has to survive by time τ_{n+1}.

By the preliminary assumption (58), the first term at the r.h.s. converges to 0 as $n \to \infty$, whereas the second one tends to 0 by (54). Concerning the third, conditional expectation term, estimate the interior probability from above by using the definition of p_n given before (55), and its estimate (57) to obtain the bound

$$P_{\tau_n,\lambda_n\delta_{2^{-n}a}}\big(X_{\tau_{n+1}} \ne 0\big) \le 1 - p_n \le 1 - \exp\!\big[-\lambda_n v(a)\big] \le \lambda_n v(a) \le \varepsilon_n v(a).$$

But the latter expression bounds the total third term at the r.h.s. of (59) and converges to 0 as $n \to \infty$.

Consequently, (50) is true, provided we know (58) which is all that remains to be shown.

Step 4° In order to prove (58) we intend to estimate the probability expression in (58) from above by a term converging to 0 as $n \to \infty$. For the purpose of getting such an estimate, we will *fix* an n, and we will study $\widetilde{X}$ only until τ_n, that is until the particles reach $2^{-n}a$. Until this time, we may read our hyperbolic branching rate ϱ_2 as a truncated rate $\varrho_2 \wedge K$, for a suitable K we fix from now on.

Step 5° We next intend to define a *new time scale* denoted by r. Given for the moment $w \in \mathbf{C}_a$, set

$$R(t) := \int_0^t \mathrm{d}s \, \bigl[\varrho_2(w_s) \wedge K\bigr], \qquad t \geq 0. \tag{60}$$

Note that with Π_a–probability one, $R(t) \to \infty$ as $t \uparrow \infty$. Since $t \mapsto R(t)$ is strictly increasing Π_a–a.s., define finite stopping times $\sigma(r)$ (converging to infinity as $r \to \infty$) by $R\bigl(\sigma(r)\bigr) = r,\ r \geq 0$. Note that

$$\tfrac{\mathrm{d}\sigma(r)}{\mathrm{d}r} = \tfrac{1}{\varrho_2(w_{\sigma(r)}) \wedge K} \tag{61}$$

for almost all r. Therefore the time change to the scale r will cancel the branching rate $\varrho_2 \wedge K$, for instance in the variance formula (47). Basically this will enable us to use the well-know fact, that the total mass process of the continuous super-Brownian motion with uniform branching rate satisfies the stochastic equation (63) below.

Step 6° Define $Y_n := \widetilde{X}^K_{\tau_n}(E_n)$ and set

$$Z_r := \bigl\langle \widetilde{X}^K_{\sigma(r)}, 1 \bigr\rangle, \qquad r \geq 0. \tag{62}$$

Assume for the moment that under the probability law $\widetilde{P}^K_a$ the following two statements hold:

(i) If $Z_{\xi_n} = 0$ then $Y_n = 0$.

(ii) The process Z satisfies

$$\mathrm{d}Z_r = \sqrt{2Z_r}\,\mathrm{d}B_r\,, \quad Z_0 = 1, \tag{63}$$

for some Brownian motion B.

Then,

$$\widetilde{P}^K_a\left(\widetilde{X}^K_{\tau_n}(E_n) > 0\right) = \widetilde{P}^K_a\,(Y_n > 0) \leq \widetilde{P}^K_a\bigl(Z_{\xi_n} > 0\bigr),$$

and from the well-know survival probability formula for solutions Z of (63), that is of Feller's branching diffusion, we continue with $= 1 - \mathrm{e}^{-1/\xi_n} \leq 1/\xi_n$. Since the fixed n was arbitrary, we can let n tend to ∞ to arrive at (58).

Step 7° We are left with proving the statements **(i)** and **(ii)**. The first one is easy. Indeed, $\langle \widetilde{X}^K_{\sigma(\xi_n)}, 1\rangle = Z_{\xi_n} = 0$ means that paths which reach an accumulated rate of branching ξ_n have died. But paths in E_n have an accumulated rate greater than or equal to ξ_n. Hence they must also be died: $Y_n = \widetilde{X}^K_{\tau_n}(E_n) = 0$.

Now we can concentrate on proving **(ii)**. The initial condition is trivially fulfilled. Following Dynkin's terminology, we let $\mathcal{G}_{\sigma(r)},\ r \geq 0$, denote the pre–$\sigma(r)$ σ–fields appearing in the formulation of the special Markov property of the (truncated) stopped historical superprocess $\widetilde{X}^K$. It suffices to show that Z is a $\big(\widetilde{P}^K_a, \mathcal{G}_{\sigma(r)}\big)$–martingale with square variation

$$\langle\!\langle Z\rangle\!\rangle_r \ = 2\int_0^r \mathrm{d}s\ Z_s\,, \qquad r \geq 0.$$

This statement would be verified if we demonstrated that for $0 \leq r < r'$,

$$\left.\begin{aligned} \widetilde{P}^K_a\big(Z_{r'}\,\big|\,\mathcal{G}_{\sigma(r)}\big) &= Z_r\,, \\ \widetilde{P}^K_a\Big(Z^2_{r'} - 2\textstyle\int_r^{r'}\mathrm{d}s\ Z_s\Big|\,\mathcal{G}_{\sigma(r)}\Big) &= Z^2_r\,. \end{aligned}\right\} \tag{64}$$

Now we claim that it is enough to show that for each fixed $T > 0$ and finite measure μ on $\mathbb{R}$

$$\widetilde{P}^K_{0,\mu}\, Z_T \ = \ \langle \mu, 1\rangle\,, \tag{65}$$

$$\widetilde{P}^K_{0,\mu}\Big(Z^2_T - 2\textstyle\int_0^T \mathrm{d}s\ Z_s\Big) \ = \ \langle \mu, 1\rangle^2\,. \tag{66}$$

In fact, by projection as in (39), Z_r of (62) coincides with $\big\langle X^K_{\sigma(r)}, 1\big\rangle$. Thus, indeed we can use the time-homogeneity of the super-Brownian motion X^K, in conjunction with Dynkin's special Markov property to see that the statements (65) and (66) are sufficient for (64).

Step 8° It remains to prove the identities (65) and (66). The expectation formula (65) directly follows from (46). Using this, the statement (66) is equivalent to

$$\widetilde{\mathrm{V}}\mathrm{ar}^K_{0,\mu}\, Z_T \ = \ 2\,T\,\langle \mu, 1\rangle\,. \tag{67}$$

But from the variance formula (47) we get

$$\widetilde{\mathrm{V}}\mathrm{ar}^K_{0,\mu}\, Z_T \ = \ \widetilde{\mathrm{V}}\mathrm{ar}^K_{0,\mu}\big\langle \widetilde{X}^K_{\sigma(T)}, 1\big\rangle \ = \ 2\,\widetilde{\Pi}_{0,\mu}\int_0^{\sigma(T)} \mathrm{d}s\ \big(\varrho_2 \wedge K\big)(W_s).$$

Substituting $s = \sigma(r)$ and recalling (61), we arrive at (67).

This completes the proof of Theorem 23. ∎

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